Efficient value-at-risk estimation for mortgage-backed securities

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We develop an efficient Monte Carlo simulation-based methodology for value-at-risk (VAR) and sensitivity analysis of mortgage-backed securities (MBS) that employs an importance sampling technique developed for quadratic VAR models. Our approach, whose validity is derived from a fundamental result in perturbation analysis, is applicable to any analytic interest rate and prepayment model, and more generally to any path-dependent cashflows that admit analytic gradients. We compare the accuracy and computational performance of our VAR estimators with those obtained via finite-difference gradient approximation schemes.

1 INTRODUCTION

Cashflow analysis of mortgage-backed securities (MBS) continues to represent one of the major challenges in financial engineering. The complexity of MBS cashflows can be traced to the complicated and intricate relationship between prepayment behavior and the interest rate term structure – the cashflows are stochastic and path-dependent. Leaving aside the choice of interest rate model, and the difficulties behind constructing an accurate prepayment model, which has evolved into a cottage industry in itself, at present it is generally accepted that Monte Carlo simulation remains the only viable recourse for realistic analysis of MBS cashflows. It is also true that despite the many recent advances in improving the speed and accuracy of Monte Carlo simulations (see, eg, Glasserman (2004)), Monte Carlo methods remain a heavily computation-intensive procedure.

In a recent paper, Chen and Fu (2002) proposed a Monte Carlo method for simultaneously evaluating the price and sensitivities (the so-called greeks) for fixed- and adjustable-rate pass-through MBS. Assuming a simple analytic prepayment model and a one-factor Hull–White interest rate model, they show through
simulation studies that improvements of up to 70% in computational efficiency can be attained over standard methods that rely on finite-difference derivative approximations for estimating the sensitivities (we henceforth refer to the latter category of methods as finite-difference methods). These gains can be traced to the fact that in their MBS pricing formulation the expectation and differentiation operators commute – this is a simple consequence of the various smoothness assumptions behind the models for prepayment and the interest rate term structure, but it can also be rigorously justified via perturbation analysis (PA) (Ho and Cao, 1991).

The two approaches – PA versus finite-difference – represent the classical tradeoff between the number of iterations versus the computational complexity per iteration: in the PA-based approach, one set of trials is sufficient to compute the price and first- and second-order sensitivities. On the other hand, each forward path evaluation is computationally quite involved, and requires a priori differentiation of the prepayment and term structure models, a far from trivial task. The alternative is to use finite-difference schemes to compute the sensitivities, but the obvious drawbacks are that a far greater number of simulations are required and that the results are generally less accurate.

Motivated by the results of Chen and Fu (2002), the primary aim of this paper is to develop an efficient Monte Carlo methodology for the evaluation of value-at-risk (VAR) for mortgage-backed securities. The only previous work addressing MBS VAR estimation that we are aware of is Jakobsen (1996), which takes an altogether different approach based on mapping MBS into the RiskMetrics framework. The case study of Chen and Fu only addresses the MBS price and some basic sensitivities. The extension to VAR is not only computationally more challenging but also involves a number of subtle but important modeling choices that impact both accuracy and computational efficiency. Our main objective therefore is to construct the appropriate Monte Carlo framework for MBS VAR that balances these multiple needs.

Any Monte Carlo method for estimating MBS VAR requires, among other things, selecting a VAR methodology that balances the computational requirements with the necessary accuracy. One has at one’s disposal methods ranging from straightforward linear (delta-normal) to quadratic (delta–gamma) models, as well as Monte Carlo-based approaches. Also, in evaluating sensitivities through either finite-difference schemes or our PA-based approach, the variance of the simulation turns out to be a critical issue – even more so when higher-order derivatives are involved. Increasing the number of paths in a simulation is not always likely to lead to desired improvements in precision. In our approach we apply an importance sampling technique for quadratic VAR models, based on methods initially developed in Glasserman, Heidelberger and Shahabuddin (1999) for the Heath–Jarrow–Morton term structure framework.

Other factors that affect our MBS VAR framework are the choice of interest rate and prepayment models. Despite the various assumptions and underlying uncertainties behind prepayment models, at present there seem to be no practical
alternatives to not using them, and our framework is developed in such a way that it admits any analytic prepayment model (in the sense of allowing analytic gradients with respect to the parameters of interest). For demonstrative purposes we use the Goldman Sachs prepayment model developed by Richard and Roll (1989), which, although outdated (see, eg, Hayre and Young (2001) for an alternative class of prepayment models), provides a useful computational benchmark for more sophisticated prepayment models.

With respect to the choice of interest rate model, as is well known, the perfect correlation that exists between long and short rates for a one-factor model (such as the Hull–White model used in Chen and Fu (2002)) can limit their effectiveness when modeling complex instruments such as MBS, where both short and long rates impact the price in a non-trivial way. For example, Chan and Russell (2001) state that the two dominant factors affecting US mortgage rates are the 10-year Treasury rate, and to a lesser but still significant extent the spread between the 10-year rate and shorter rates. In the two-factor Salomon Smith Barney interest rate model developed by Chan and Russell (2001), dependence on these two factors is represented by defining spot volatility parameters for both the rate and the slope of the short end of the yield curve.

For our purposes we develop our framework around the G2++ formulation of the widely used two-factor Hull–White interest rate model; the original Hull–White model is in fact structurally quite similar to the model of Chan and Russell (2001). We show that tractable analytic formulas for the gradient estimators can be obtained with only a modest increase in complexity. We emphasize again, however, that any interest rate model that admits analytic bond price formulas can be used in our methodology in lieu of the Hull–White two-factor model.

The second objective of this paper is to more accurately assess the computational performance of our proposed Monte Carlo estimators for MBS VAR. By estimating VAR for a set of fixed and adjustable rate pass-through MBS, using a two-factor interest rate model and a representative analytic prepayment model, we can get a reasonable picture of the computational requirements for more exotic MBS products (provided analytic cashflow expressions are available) that rely on more sophisticated interest and prepayment models. Through simulation studies we assess the advantages and disadvantages of the PA-based approach vs. finite-difference methods and the computational benefits of importance sampling.

Beyond resolving these technical questions, however, our results confirm that the devil truly is in the details. In some sense the fundamental idea underlying our approach is fairly simple: select analytically tractable models for the interest rate and prepayment behavior, derive analytic gradients and Hessians for the desired quantities, and adapt any of a number of speed-up techniques in the Monte Carlo simulation. In principle this methodology can be extended to any stochastic, path-dependent cashflow (eg, collateralized debt obligations). Our results emphasize the importance of choosing models that are analytically manageable and consistent with each other and which balance the competing needs of accuracy and computational efficiency.
The paper is organized as follows. Section 2 describes our basic Monte Carlo framework for sensitivity and delta–gamma VAR analysis. Section 3 provides details of the importance sampling procedure. Section 4 presents the results of our numerical study, while Section 5 suggests directions for further refinement of this methodology to MBS, and more generally to the VAR analysis of other random cashflows.

2 VALUE-AT-RISK AND SENSITIVITY ANALYSIS OF MBS

In this section we describe the general Monte Carlo simulation framework for MBS VAR estimation. As is well known, the price of any security can be written as the net present value of its discounted cashflow. In what follows we assume the closed time interval \([0,T]\) (unless otherwise noted, throughout this paper we follow common practice and express time in years, i.e., \(T = 1\) corresponds to one year) sampled at \(M\) intervals, with time step \(\Delta t\); the discrete times are denoted by \(t_k = k \Delta t\), \(k = 0,\ldots,M\). The security pricing formula is given by

\[
P = \mathcal{E}\{V\} = \mathcal{E}\left\{ \sum_{k=1}^{M} d[k]c[k] \right\},
\]

where \(\mathcal{E}\{\cdot\}\) denotes expectation, \(P\) is the price, \(V\) is the (random) value of the security, \(d[k]\) is the discount factor at time \(t_k\), and \(c[k]\) is the payoff at time \(t_k\). By the strong law of large numbers,

\[
\mathcal{E}\{V\} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} V_i
\]

where \(V_i\) is the value associated with path \(i\).

The discount factor \(d[k]\) is evaluated from the short-rate process:

\[
d[k] = d[0,1] \cdots d[k-1,k] = \prod_{i=0}^{k-1} e^{-r[i]\Delta t} = \exp\left( -\Delta t \sum_{i=0}^{k-1} r[i] \right)
\]

where \(d[i,i+1]\) is the discount factor for the end of period \(i+1\) at the end of period \(i\), and \(r[i]\) denotes the short rate observed at the end of period \(i\). Once a continuous interest rate model is chosen and appropriately discretized, \(d[k]\) becomes available for every forward path generated for the discretized short-rate process.

If \(P\), the price of the security, is a continuous function of the parameters of interest, which we henceforth denote \(\theta\), we then obtain the following gradient estimator by differentiating both sides of Equation (1):

\[
\frac{\partial P(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \mathcal{E}\{V(\theta)\} = \mathcal{E}\left\{ \sum_{k=1}^{M} \frac{\partial}{\partial \theta} (d[k,\theta]c[k,\theta]) \right\}
\]
where
\[
\frac{\partial}{\partial \theta} (d[k, \theta]c[k, \theta]) = \frac{\partial d[k, \theta]}{\partial \theta} c[k, \theta] + d[k, \theta] \frac{\partial c[k, \theta]}{\partial \theta}
\] (5)

For notational convenience we henceforth suppress the \( \theta \) argument in all the variables.

Results from perturbation analysis provide the technical conditions under which the expectation and differentiation operators commute; in our case smoothness with respect to the parameters of interest suffices. Under this smoothness assumption, the gradients of \( P \) can be evaluated according to Equation (4). The original problem of gradient estimation therefore reduces to the estimation, at each time step, of the two gradients \( \partial c/\partial \theta \) and \( \partial d/\partial \theta \).

The formula for the second-order gradient is given by
\[
\frac{\partial^2 P(\theta)}{\partial \theta^2} = \mathcal{E} \left\{ \sum_{k=1}^{M} \frac{\partial^2 d[k]}{\partial \theta^2} c[k] + 2 \frac{\partial d[k]}{\partial \theta} \frac{\partial c[k]}{\partial \theta} + \frac{\partial^2 c[k]}{\partial \theta^2} d[k] \right\}
\] (6)

Note that \( \theta \) can be a vector of the parameters of interest, in which case \( \partial^2 P(\theta)/\partial \theta^2 \) denotes the Hessian matrix.

### 2.1 Estimating value-at-risk

The change of a security price with respect to the changes in the value of risk factors can be approximated by Taylor series expansion:\(^1\)
\[
\Delta P(\theta) = \delta^T \Delta \theta + \frac{1}{2} \Delta \theta^T \Gamma \Delta \theta + \ldots
\] (7)

where \( \theta \) denotes the vector of risk factors, \( \delta = \partial P/\partial \theta \) and \( \Gamma = \partial^2 P/\partial \theta^2 \). The \( i \)th component of the gradient vector is given by
\[
\delta_i = \frac{\partial P}{\partial \theta_i} = \mathcal{E} \left\{ \sum_{k=1}^{M} \frac{\partial}{\partial \theta_i} (d[k]c[k]) \right\}
\] (8)

and the \( ij \)th component of the Hessian matrix is given by
\[
\Gamma_{ij} = \frac{\partial^2 P}{\partial \theta_i \partial \theta_j} = \mathcal{E} \left\{ \sum_{k=1}^{M} \frac{\partial^2 d[k]}{\partial \theta_i \partial \theta_j} c[k] + \frac{\partial d[k]}{\partial \theta_i} \frac{\partial c[k]}{\partial \theta_j} + \frac{\partial d[k]}{\partial \theta_j} \frac{\partial c[k]}{\partial \theta_i} + \frac{\partial^2 c[k]}{\partial \theta_i \partial \theta_j} d[k] \right\}
\] (9)

\(^1\) The drift term, \( (\partial P/\partial \theta) \Delta t \), is sometimes considered in the formula, but the effect is usually small for short horizons and is assumed zero here.
The VAR measure derived from the linear approximation is referred to as delta-normal VAR, while that derived from the second-order (quadratic) approximation is referred to as delta–gamma VAR.

Since the primary source of market risk related to MBS is interest rate risk, we accordingly define the risk factors as the spot rates for pre-specified maturities, i.e.,

\[ \theta = [R(0, t_1) R(0, t_2) \cdots R(0, t_n)]^T \]

where \( R(0, t_i) \) is the spot rate at \( t = 0 \) for maturity \( t_i \). Assuming the spot rates are normally distributed, the price of a security whose value is a function of the spot rates also follows a normal distribution under the first-order approximation

\[ \Delta P \sim N(0, \delta^T \Sigma_R \delta) \]  \hspace{1cm} (10)

where \( \Sigma_R \) denotes the covariance matrix of the risk factors. Thus, once the gradients of \( P \) and the covariance matrix \( \Sigma_R \) are known, VAR can be readily computed by multiplying the standard deviation of \( \Delta P \) by an appropriate coefficient, e.g., 1.96 for 95% VAR.

When the present value of a security depends non-linearly on the risk factors, delta–gamma VAR generally provides a better approximation to the true VAR;\(^2\) Britten-Jones and Schaefer (1999), Cárdenas et al. (1997) and Rouvinez (1997) are good sources for the derivation and application of this approach. They all formulate the price change as a linear polynomial of independent random variables but employ different numerical methods to compute the distribution of the price. We now briefly review the calculation of the delta–gamma VAR.

Define \( y \) such that \( Cy = \Delta \theta \), where \( C \) is a matrix satisfying \( CC^T = \Sigma_R \). Equation (7) is rewritten under the quadratic approximation as

\[ \Delta P = \delta^T Cy + \frac{1}{2} y^T C^T \Gamma C y \]  \hspace{1cm} (11)

Using spectral decomposition, we get

\[ C^T \Gamma C = D \Lambda D^T \]  \hspace{1cm} (12)

where \( D \) is the orthogonal matrix composed of the eigenvectors of \( C^T \Gamma C \) and \( \Lambda \) is the diagonal matrix containing the eigenvalues of \( C^T \Gamma C \). Substituting Equation (12) into (11), we have

\[ \Delta P = b^T x + \frac{1}{2} = x^T \Lambda x \]  \hspace{1cm} (13)

where \( x = D^T y = D^T C^{-1} \Delta \theta \) and \( b = D^T C^T \delta \). Observe that \( x \) follows the standard

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\(^2\) This is not always the case. If the security is a convex or concave function of the risk factors, delta–gamma VAR performs well, but it does not necessarily guarantee better performance than delta-normal VAR. See Britten-Jones and Schaefer (1999) for a detailed discussion.
normal distribution, ie, \( x \sim N(0, I) \). Assuming first \( m \) eigenvalues are non-zero, Equation (13) can be rewritten as

\[
\Delta P = \sum_{k=1}^{m} \gamma_k Q_k + \beta Q_0 + \alpha
\]  

(14)

where

\[
Q_k \sim \chi^2 \left(1, d_k^2\right), \quad d_k = \frac{b_k}{\Lambda_{kk}}, \quad k > 0, \\
Q_0 \sim N(0,1), \\
\gamma_k = \frac{1}{2} \Lambda_{kk}, \\
\beta = \sqrt{\sum_{k=m+1}^{n} b_k^2}, \\
\alpha = -\sum_{k=1}^{m} \frac{1}{2} \frac{b_k^2}{\Lambda_{kk}}
\]

That is, \( \Delta P \) is expressed as a linear polynomial of a standard normal random variable and (assuming non-zero eigenvalues) independent non-central \( \chi^2 \) random variables. The characteristic function of \( \Delta P \) has the following form (Holton, 2003):

\[
\Psi(w) = \exp \left( i w \alpha - \frac{w^2 \beta^2}{2} + i w \sum_{k=1}^{m} \frac{\gamma_k d_k^2}{1 - 2 i w \gamma_k^2} \right) \\
\prod_{k=1}^{m} \frac{1}{\sqrt{1 - 2 i w \gamma_k^2}}
\]  

(15)

By the inversion theorem, the cumulative distribution function of \( \Delta P \) is given by

\[
\Phi(\Delta P) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{A} \sin(B + C)}{D} \, dw
\]  

(16)

where

\[
A = -\frac{w^2}{2} \left( \beta^2 + 4 \sum_{k=1}^{m} \frac{\gamma_k^2 d_k^2}{1 + 4 \gamma_k^2 w^2} \right)
\]  

(17)

\[
B = w \left( \alpha - \Delta P + \sum_{k=1}^{m} \frac{\gamma_k d_k^2}{1 + 4 \gamma_k^2 w^2} \right)
\]  

(18)

\[
C = \frac{1}{2} \sum_{k=1}^{m} \arctan \left( 2 \gamma_k w \right)
\]  

(19)

\[
D = w \left( \prod_{k=1}^{m} \left( 1 + 4 \gamma_k^2 w^2 \right) \right)^{1/2}
\]  

(20)

A numerical algorithm can be employed to calculate the cumulative distribution function in (16); \( \text{VAR} (\Delta P) \) for a specific confidence level can now be found via optimization.
2.2 Sensitivity analysis

We next consider sensitivity of the price with respect to changes in the initial yield curve. The most common assumption for yield curve changes is a parallel shift over all maturities. Such changes, while the most dominant, do not capture the range of possible yield curve changes. One means of systematically modeling such diverse changes, proposed by Chen and Fu (2002), is to express the yield curve shifts in exponential harmonic form:

\[ \Delta R(0,t) = \sum_{m=0}^{\infty} \Delta_m \cos\left(m\pi\left(1 - e^{-t/T_0}\right)\right) \]  

(21)

where \( T_0 \) is a user-specified parameter defining the harmonic shift; a small value of \( T_0 \) implies that short rates and long rates will move with a greater degree of independence. As \( T_0 \) becomes larger the exponential parameterization approaches the standard harmonic basis function.

In practice it is not uncommon to see prices affected more by changes in the short-term rate rather than the long-term rate (or vice versa, depending on the security). Using the exponential harmonic series effectively places a greater emphasis on changes in the spot rate for maturities less than \( T_0 \). We thus take the perturbed spot rate function to be of the form

\[ R'(0,t) = R(0,t) + \sum_{m=0}^{\infty} \Delta_m \cos\left(m\pi\left(1 - e^{-t/T_0}\right)\right) \]  

(22)

The gradients of the price with respect to \( \Delta_m \) measure the sensitivity of the price to the yield curve change described by the \( m \)th harmonic function. Indeed, the duration and convexity are essentially normalized versions of the first- and second-order gradients with respect to \( \Delta_0 \); for this reason the terms duration and convexity are used to mean the first- and second-order gradients with respect to any parameters of interest included in \( \theta \):

\[ \text{Duration} = -\frac{\partial P(\theta)}{\partial \theta} \frac{1}{P(\theta)} \]  

(23)

\[ \text{Convexity} = \frac{\partial^2 P(\theta)}{\partial \theta^2} \frac{1}{P(\theta)} \]  

(24)

For example, price sensitivity to yield volatility is also of interest for non-linear securities; this sensitivity is addressed later.

3 IMPORTANCE SAMPLING

In this section, we outline the importance sampling method for our VAR estimation procedure. For our purposes we adopt the procedure given by Glasserman,
Heidelberger and Shahabuddin (1999), which combines importance sampling with stratified sampling to price options under the Heath–Jarrow–Morton framework; further details are given in Glasserman (2004).

Consider a general pricing formula of the form

\[ P = \mathcal{E}\{V(Z)\} = \int_{\mathbb{R}^n} V(z) \phi_n(z) \, dz \]  \hspace{1cm} (25)

where \( V(Z) \) is the value of the security given input \( Z \), which in our model has the form in Equation (1), \( Z \) is a normally distributed \( n \)-dimensional random vector, and \( \phi_n(z) \) is the standard \( n \)-dimensional zero-mean, unit variance normal density. Equation (25) is rewritten in the form

\[ P = \int_{\mathbb{R}^n} V(z) \frac{\phi_n(z)}{\psi(z)} \psi(z) \, dz \]  \hspace{1cm} (26)

for any positive density function \( \psi(z) \). Estimation of the price using Monte Carlo simulation under \( \psi(z) \) proceeds according to

\[ \hat{P} = \frac{1}{N} \sum_{i=1}^{N} V(Z^{(i)}) \frac{\phi_n(Z^{(i)})}{\psi(Z^{(i)})} \]  \hspace{1cm} (27)

where \( Z^{(i)} \), \( i = 1, \ldots, N \), are independently drawn samples from \( \psi(z) \). The key idea behind importance sampling is to find \( \psi(z) \) such that more samples are drawn from important regions, thereby reducing the variance of the simulation. Glasserman, Heidelberger and Shahabuddin (1999) restrict \( \psi(z) \) to \( \phi_n(z - \mu) \), where \( \mu \) solves the following optimization:

\[ \mu = \arg \max_{z \in \mathbb{R}^n} V(x) \phi_n(z) \]  \hspace{1cm} (28)

The implication of this approach is that the zero-variance density \( V(z) \phi_n(z) \) is approximated by a normal density whose mode coincides with that of the optimal density, and whose covariance matrix coincides with that of the original normal density. Once \( \psi(z) \) is obtained, stratified sampling can be applied. For completeness we repeat the steps for stratified sampling as given in Glasserman, Heidelberger and Shahabuddin (1999):

1. Draw \( N \) independent samples, \( U^{(i)} \), \( i = 1, \ldots, N \), from a uniform distribution over \((0, 1)\).
2. Set

\[ V^{(i)} = \frac{i - 1}{N} + \frac{U^{(i)}}{N}, \hspace{1cm} i = 1, \ldots, N \]
3. Generate normally distributed \( N \) independent samples, \( X^{(i)} \):

\[ X^{(i)} = \phi^{-1}(V^{(i)}) \]

where \( \phi^{-1}(\cdot) \) is the inverse of the standard normal distribution function.

4. Draw \( Y^{(i)}, i = 1, \ldots, N \), from \( N(0, I_N) \), where \( I_N \) is an \( N \times N \) identity matrix.

5. Set

\[ Z^{(i)} = uX^{(i)} + (I_n - uu')Y^{(i)} + \mu \]

where \( u \) is the stratification direction, which can be obtained either from the eigenvectors of the Hessian for \( \log V(z) \) at \( \mu \), or simply approximated by \( \mu \).

6. Evaluate the likelihood ratios:

\[ L^{(i)} = \exp\left(-\mu'Z^{(i)} + \frac{1}{2}uu'\mu\right) \]

7. Estimate the price:

\[ \hat{P} = \frac{1}{N} \sum_{i=1}^{N} V(Z^{(i)})L^{(i)} \]

The importance sampling technique outlined here has the overhead of solving an eigenvalue problem as well as an optimization; approximation methods for estimating \( \mu \) and \( u \) are further proposed by Glasserman, Heidelberger and Shahabuddin (1999).

In our case, the optimal drift vector \( \mu \) obtained for the price is not necessarily optimal for its derivatives. Numerical studies further indicate that the associated objective function for \( \mu \) is poorly behaved for some derivatives as the optimization failed to converge in some limited cases. In our approach we therefore use the same value of \( \mu \) obtained from the price for each of the derivatives. This is not as serious as it seems; while it is unlikely that the importance regions for the price and derivatives coincide, our numerical studies indicate that importance sampling with the optimal drift vector for the price also reduces the variances for the derivative estimates. Numerical results are provided below.

### 4 NUMERICAL STUDY

In this section, we apply our gradient estimators developed in the previous sections to estimate VAR and sensitivities of a fixed-rate pass-through mortgage-backed security (FRM) and an adjustable rate pass-through mortgage-backed security (ARM). The results are compared to those obtained via the finite-difference method and the overall computational performance of our VAR estimators is assessed.
4.1 Prepayment model

For our numerical study we use the Goldman Sachs prepayment model, developed by Richard and Roll (1989) for predicting mortgage prepayment behavior. It assumes a general prepayment function of the form

$$C_{pr}[k] = RI[k] \cdot AGE[k] \cdot MM[k] \cdot BN[k]$$  \hspace{1cm} (29)

where $RI[k]$ denotes the refinancing incentive, $AGE[k]$ denotes the seasoning multiplier, $MM[k]$ is the monthly multiplier (constant over any given month), and $BN[k]$ denotes the burnout multiplier:

$$RI[k] = C_1 \arctan \left\{ -C_2 + C_3 \left[ G - R(t_k, t_{k+\tau}) \right] \right\} + C_4$$  \hspace{1cm} (30)

$$AGE[k] = \min \left( 1, \frac{t_k}{T} \right), \text{ for some } T < \text{ loan maturity}$$  \hspace{1cm} (31)

$$MM[k] = \text{ monthly multiplier}$$  \hspace{1cm} (32)

$$BM[k] = C_5 + (1 - C_5) \frac{B[k-1]}{B[0]}$$  \hspace{1cm} (33)

where $R(t_k, t_{k+\tau})$ denotes the spot rate at time $t_k$ for maturity date $t_{k+\tau}$, and $C_5$ is a constant coefficient between 0 and 1. The various coefficients and monthly multipliers can be estimated from historical data by optimizing a suitable non-linear least-squares fitting criterion.

4.2 Interest rate model

For our numerical study we use the two-factor G2++ formulation in Brigo and Mercurio (2001) of Hull and White’s two-factor short-rate model:

$$dx(t) = -\alpha x(t) \, dt + \gamma \, dw_1(t)$$  \hspace{1cm} (34)

$$dy(t) = -\beta y(t) \, dt + \eta \, dw_2(t)$$  \hspace{1cm} (35)

$$\phi(t) = f(0,t) + \frac{\gamma^2}{2\alpha^2} \left( 1 - e^{-2\alpha t} \right)^2 + \frac{\eta^2}{2\beta^2} \left( 1 - e^{-2\beta t} \right)^2$$

$$+ \kappa \frac{\gamma \eta}{\alpha \beta} \left( 1 - e^{-\alpha t} \right) \left( 1 - e^{-\beta t} \right)$$  \hspace{1cm} (36)

$$r(t) = x(t) y(t) + \phi(t)$$  \hspace{1cm} (37)

where $dw_1(t) \cdot dw_2(t) = \kappa \, dt$ and $x(0) = y(0) = 0$.

Assume that the time interval of interest is given by $[0, T]$. Given that $(x(t), y(t))$ is a two-dimensional Ornstein–Uhlenbeck process, we adopt the following iteration:

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\[ x[k+1] = -e^{-\alpha \Delta t} x[k] + \gamma \sqrt{\frac{1-e^{-2\alpha \Delta t}}{2\alpha}} z_1[k+1] \]  
\[ y[k+1] = -e^{-\beta \Delta t} y[k] + \eta \sqrt{\frac{1-e^{-2\beta t}}{2\beta}} z_2[k+1] \]  
\[ \phi[k] = f(0, t_k) + \frac{\gamma^2}{2\alpha^2} (1 - e^{-2\alpha t_k})^2 + \frac{\eta^2}{2\beta^2} (1 - e^{-2\beta t_k})^2 \]
\[ + \kappa \frac{\gamma \eta}{\alpha \beta} (1 - e^{-\alpha t_k}) (1 - e^{-\beta t_k}) \]  
\[ r[k] = x[k] + y[k] + \phi[k] \]

where \( t_k = k \Delta t, f(0, t_k) = R(0, t_k) + t_k (\partial / \partial t) R(0, t) \big|_{t=t_k} \) is the instantaneous forward rate, and \((z_1[k], z_2[k]) \sim N(0, Q)\) with
\[ Q = \begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix} \]  

For the discretized short-rate dynamics, the formula for the discount factor is given by
\[ d[k] = \exp\left(-\Delta t \sum_{i=0}^{k-1} r[i]\right) \]  

Its first-order gradient with respect to a generic parameter \( \theta \) is given by
\[ \frac{\partial d[k]}{\partial \theta} = -d[k] \Delta t \sum_{i=0}^{k-1} \frac{\partial r[i]}{\partial \theta} \]  

Detailed formulas for all gradients will be made available upon request.

### 4.3 Cashflows and their gradients

We adopt the cashflow model for mortgage pass-throughs as described in Chapter 25 of Fabozzi (2001). We consider the case where payments are made on a monthly basis over a total of \( M \) months; in the absence of prepayments the monthly proceeds are fixed (ie, level payments). The formula for the cashflow is
\[ c[k] = P_s[k] + I[k] + P_r[k] \]  

where \( c[k] \) denotes the payoff in month \( k \), \( P_s[k] \) and \( I[k] \) respectively denote the principal and interest portions of the scheduled monthly payment, and \( P_r[k] \) denotes
the unscheduled principal payment (the prepayment amount). The remaining (end-of-month) balance for month \( k \), \( B[k] \), is obtained from the iteration

\[
B[k] = B[k-1] - P_s[k] - P_r[k]
\] (46)

The formula for \( P_r[k] \) is

\[
P_r[k] = (B[k-1] - P_s[k]) S_{mm}[k]
\] (47)

where \( S_{mm}[k] \) denotes the single monthly mortality. It is usual to express prepayment rates in terms of the annualized conditional prepayment rate \( C_{pr}[k] \), in which case

\[
S_{mm}[k] = 1 - (1 - C_{pr}[k])^{\frac{1}{12}}
\] (48)

If \( \theta \) is a generic parameter, then

\[
\frac{\partial c[k]}{\partial \theta} = \frac{\partial B[k-1]}{\partial \theta} (A[k]1 - S_{mm}[k]) + C[k] \cdot S_{mm}[k] + \frac{\partial S_{mm}[k]}{\partial \theta} B[k-1](-A[k] + C[k])
\] (49)

where

\[
A[k] = \frac{G[k]/12}{1 - \left(\frac{1+G[k]}{12}\right)^{M-k+1}}
\] (50)

\[
C[k] = \left(1 + \frac{G[k]}{12}\right)
\] (51)

\( G[k] \) is the gross coupon rate at \( k \) and \( M \) is the maturity in months. The gradient of \( B[k] \) can be computed recursively via

\[
\frac{\partial B[k]}{\partial \theta} = C[k] \frac{\partial B[k-1]}{\partial \theta} - \frac{\partial c[k]}{\partial \theta}
\] (52)

with an initial condition

\[
\frac{\partial B[0]}{\partial \theta} = 0
\] (53)

The gradient of \( S_{mm}[k] \) is given by

\[
\frac{\partial S_{mm}[k]}{\partial \theta} = \frac{1}{12} \left(1 - C_{pr}[k]\right)^{-\frac{11}{12}} \frac{\partial C_{pr}[k]}{\partial \theta}
\] (54)

Explicit formulas for all gradients will be made available upon request.
### 4.4 Simulation setup

The data assumed for all the simulations are as shown below:

- **Fixed-rate MBS**
  - Maturity: 30 years (360 time steps)
  - Coupon: 5.0% fixed, paid monthly
  - Initial balance: $10,000,000

- **Adjustable-rate MBS**
  - Maturity: 30 years (360 time steps)
  - Coupon: Indexed by Treasury one-year rate, adjusted yearly, paid monthly
  - Margin: 0.5%
  - Period cap/floor: \( t - 1 \) coupon rate ± 2%
  - Life cap/floor: 10%/none
  - Initial balance: $10,000,000

- **Prepayment parameters**

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( \tau )</th>
<th>( T )</th>
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<tbody>
<tr>
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<td>8.571</td>
<td>430</td>
<td>0.28</td>
<td>0.3</td>
<td>3</td>
<td>2.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Jan</th>
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<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
</tr>
</thead>
<tbody>
<tr>
<td>( MM )</td>
<td>0.94</td>
<td>0.76</td>
<td>0.74</td>
<td>0.95</td>
<td>0.98</td>
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</tbody>
</table>

<table>
<thead>
<tr>
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<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>( MM )</td>
<td>0.98</td>
<td>1.10</td>
<td>1.18</td>
<td>1.22</td>
<td>1.23</td>
</tr>
</tbody>
</table>

- **Interest rate model**

<table>
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<tr>
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<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \eta )</th>
<th>( \kappa )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.05</td>
<td>0.01</td>
<td>0.002</td>
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</table>

- **Initial yield curve**

<table>
<thead>
<tr>
<th>1 m</th>
<th>3 m</th>
<th>6 m</th>
<th>1 y</th>
<th>2 y</th>
<th>3 y</th>
<th>4 y</th>
<th>5 y</th>
<th>7 y</th>
<th>9 y</th>
<th>10 y</th>
<th>15 y</th>
<th>20 y</th>
<th>30 y</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>4.2</td>
<td>4.3</td>
<td>4.5</td>
<td>4.7</td>
<td>4.8</td>
<td>4.85</td>
<td>4.9</td>
<td>5.0</td>
<td>5.7</td>
<td>6.0</td>
<td>6.2</td>
<td>6.4</td>
<td>6.5</td>
</tr>
</tbody>
</table>

- **Risk factors**

We define the risk factors to be spot rates for maturities \{1 m, 3 m, 6 m, 1 y, 2 y, 3 y, 4 y, 5 y, 7 y, 9 y, 10 y, 15 y, 20 y, 30 y\}.

<table>
<thead>
<tr>
<th>1 m</th>
<th>3 m</th>
<th>6 m</th>
<th>1 y</th>
<th>2 y</th>
<th>3 y</th>
<th>4 y</th>
<th>5 y</th>
<th>7 y</th>
<th>9 y</th>
<th>10 y</th>
<th>15 y</th>
<th>20 y</th>
<th>30 y</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>4.2</td>
<td>4.3</td>
<td>4.5</td>
<td>4.7</td>
<td>4.8</td>
<td>4.85</td>
<td>4.9</td>
<td>5.0</td>
<td>5.7</td>
<td>6.0</td>
<td>6.2</td>
<td>6.4</td>
<td>6.5</td>
</tr>
</tbody>
</table>
The covariances of the risk factors are assumed as shown in the table (values are multiplied by 1.0e4):

<table>
<thead>
<tr>
<th></th>
<th>1m</th>
<th>3m</th>
<th>6m</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
<th>7y</th>
<th>9y</th>
<th>10y</th>
<th>15y</th>
<th>20y</th>
<th>30y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1m</td>
<td>1.00</td>
<td>0.87</td>
<td>0.81</td>
<td>0.74</td>
<td>0.67</td>
<td>0.60</td>
<td>0.54</td>
<td>0.48</td>
<td>0.42</td>
<td>0.37</td>
<td>0.32</td>
<td>0.26</td>
<td>0.22</td>
<td>0.18</td>
</tr>
<tr>
<td>3m</td>
<td>0.87</td>
<td>0.81</td>
<td>0.74</td>
<td>0.68</td>
<td>0.62</td>
<td>0.56</td>
<td>0.50</td>
<td>0.43</td>
<td>0.40</td>
<td>0.35</td>
<td>0.30</td>
<td>0.25</td>
<td>0.20</td>
<td>0.16</td>
</tr>
<tr>
<td>6m</td>
<td>0.81</td>
<td>0.74</td>
<td>0.72</td>
<td>0.65</td>
<td>0.60</td>
<td>0.55</td>
<td>0.49</td>
<td>0.43</td>
<td>0.38</td>
<td>0.33</td>
<td>0.28</td>
<td>0.24</td>
<td>0.20</td>
<td>0.16</td>
</tr>
<tr>
<td>1y</td>
<td>0.74</td>
<td>0.68</td>
<td>0.65</td>
<td>0.64</td>
<td>0.58</td>
<td>0.53</td>
<td>0.48</td>
<td>0.43</td>
<td>0.37</td>
<td>0.32</td>
<td>0.28</td>
<td>0.24</td>
<td>0.20</td>
<td>0.16</td>
</tr>
<tr>
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<td>0.60</td>
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<td>0.48</td>
<td>0.43</td>
<td>0.39</td>
<td>0.35</td>
<td>0.30</td>
<td>0.26</td>
<td>0.22</td>
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<td>0.16</td>
</tr>
<tr>
<td>3y</td>
<td>0.60</td>
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<td>0.55</td>
<td>0.53</td>
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<td>0.49</td>
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</tr>
<tr>
<td>4y</td>
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<tr>
<td>5y</td>
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<td>0.43</td>
<td>0.43</td>
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<td>0.37</td>
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<td>0.36</td>
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<td>0.31</td>
<td>0.30</td>
<td>0.24</td>
<td>0.21</td>
<td>0.18</td>
<td>0.15</td>
<td>0.13</td>
<td>0.13</td>
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<tr>
<td>9y</td>
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<td>0.33</td>
<td>0.32</td>
<td>0.30</td>
<td>0.28</td>
<td>0.27</td>
<td>0.25</td>
<td>0.24</td>
<td>0.21</td>
<td>0.17</td>
<td>0.15</td>
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<td>0.12</td>
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<tr>
<td>10y</td>
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<td>0.26</td>
<td>0.25</td>
<td>0.24</td>
<td>0.22</td>
<td>0.21</td>
<td>0.20</td>
<td>0.16</td>
<td>0.13</td>
<td>0.11</td>
<td>0.11</td>
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<tr>
<td>15y</td>
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<td>0.24</td>
<td>0.24</td>
<td>0.22</td>
<td>0.22</td>
<td>0.20</td>
<td>0.18</td>
<td>0.17</td>
<td>0.16</td>
<td>0.16</td>
<td>0.12</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>20y</td>
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<td>0.20</td>
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<td>0.19</td>
<td>0.18</td>
<td>0.17</td>
<td>0.16</td>
<td>0.15</td>
<td>0.15</td>
<td>0.13</td>
<td>0.12</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>30y</td>
<td>0.18</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.15</td>
<td>0.14</td>
<td>0.13</td>
<td>0.12</td>
<td>0.11</td>
<td>0.10</td>
<td>0.10</td>
<td>0.09</td>
<td>0.09</td>
</tr>
</tbody>
</table>

### 4.5 Simulation results

Average cashflow schedules for the FRM and ARM are shown in Figure 1: cashflows increase for the first three years and then decrease continuously until maturity. This is mainly because $T$ in the seasoning multiplier of the prepayment model is set to 2.5 years. The sawtooth-like fluctuations can be attributed to the monthly multiplier.

It is generally known (e.g., Chen and Fu (2002)) that close to 99% of yield curve movement is explained by the first three principal components, commonly referred as parallel shift, twist and butterfly. We therefore calculate the sensitivities of the MBS with respect to the corresponding exponential harmonic shocks $\Delta_0$, $\Delta_1$ and $\Delta_2$, with a harmonic shift $T_0 = 3$. These harmonic shocks are shown in Figure 2.

The price, gradients and Hessians are calculated using both the PA-based method and the finite-difference (FD) method, both with and without importance sampling. We compute the first- and second-order derivatives with respect to harmonic shocks and risk factors, and the first-order derivatives with respect to volatility factors. The results are summarized in Tables 1 through 4.

In the tables, the rows labelled PA(-IS) denote the estimation results from the PA-based method (with importance sampling), while FD(-IS) denote the estimation results from the finite-difference method (with importance sampling). Since the comparison results of estimators for the ARM are similar to those for the FRM, we show only a few representative results for the ARM case, denoted “ARM” in the tables. The importance sampling technique is coupled with the stratified sampling method as described earlier; standard antithetic variance
reduction is applied in the remaining cases. The drift vectors $\mu$ that optimize the objective function in (28) are shown in Figure 3. The optimal drift vector of the FRM is smooth, whereas that of the ARM has a periodic jump; the jump frequency is one year, with the size of the jump continuously decreasing. This serves as a counterexample to the observation in Glasserman, Heidelberger and Shahabuddin (1999), where it is suspected that the optimal drift vectors are smooth and might be obtained via polynomial approximation to reduce the computational burden without adversely affecting the results.

**FIGURE 1** Simulated cashflows of a fixed-rate MBS (“FRM”) and an adjustable-rate MBS (“ARM”). The horizontal line represents the level payments of the fixed-rate MBS without prepayment.

![Simulated cashflows of a fixed-rate MBS (“FRM”) and an adjustable-rate MBS (“ARM”).](image1)

**FIGURE 2** Three harmonic shocks, $\Delta_0$, $\Delta_1$ and $\Delta_2$ with harmonic shift $T_0 = 3$.

![Three harmonic shocks, $\Delta_0$, $\Delta_1$ and $\Delta_2$ with harmonic shift $T_0 = 3$.](image2)
Each simulation consists of 1,000 iterations. The values in parentheses in the tables are percentage relative errors, expressed as the standard deviation divided by the absolute average value. Relative errors are computed from 100 simulation runs.

**FIGURE 3** Optimal drift vectors, $\mu$, for importance sampling.

(a) Fixed-rate MBS

(b) Adjustable-rate MBS
4.5.1 Sensitivity analysis

The sensitivity measures, i.e., the first- and second-order derivatives with respect to the risk factors, are shown in Tables 1 through 4. The normalized versions of the gradients and Hessians can be effectively interpreted as the common risk measures, duration and convexity.

As noted earlier, importance sampling preserves the mean, and this can be confirmed by comparing the present values in Table 1. It is noteworthy that even though the simulation error of the present values is acceptably small without importance sampling, importance sampling dramatically improves precision, reducing the relative simulation error by 65%.

The first-order gradient estimates with respect to harmonic shocks, denoted Dur0, Dur1 and Dur2 in Table 1, are almost identical among the different estimators. The gradients with respect to \( \Delta_0 \) and \( \Delta_2 \) are negative as both the long- and short-term spot rates rise under these shocks, and the gradient with respect to \( \Delta_1 \) is positive as the long-term spot rates fall and the short-term spot rates rise (Figure 2); the coupon rate of the ARM is adjusted with respect to the one-year Treasury rate, so the present value of the ARM is relatively insensitive to yield curve changes.

The first-order gradients with respect to volatility factors, denoted Vega1 and Vega2, are both negative as higher volatility implies higher risk and lower price. Vega estimates have larger simulation errors compared to duration estimates due to the sensitive nature of the MBS price to the volatility factors. The simulation errors of the PA estimators tend to be consistently smaller than those of the FD

### TABLE 1

<table>
<thead>
<tr>
<th></th>
<th>PV</th>
<th>Dur0</th>
<th>Dur1</th>
<th>Dur2</th>
<th>Vega1</th>
<th>Vega2</th>
</tr>
</thead>
<tbody>
<tr>
<td>PA-IS</td>
<td>9317787</td>
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<td>59811358</td>
<td>-49985777</td>
<td>-17636198</td>
<td>-8942503</td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
<td>(0.121)</td>
<td>(0.145)</td>
<td>(0.197)</td>
<td>(1.546)</td>
<td>(4.430)</td>
</tr>
<tr>
<td>FD-IS</td>
<td>9317809</td>
<td>-68294054</td>
<td>59112524</td>
<td>-50185911</td>
<td>-17642462</td>
<td>-9072701</td>
</tr>
<tr>
<td></td>
<td>(0.013)</td>
<td>(0.128)</td>
<td>(0.156)</td>
<td>(0.208)</td>
<td>(1.561)</td>
<td>(3.636)</td>
</tr>
<tr>
<td>PA</td>
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<td>59785057</td>
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<td>-17618480</td>
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</tr>
<tr>
<td></td>
<td>(0.033)</td>
<td>(0.257)</td>
<td>(0.323)</td>
<td>(0.455)</td>
<td>(2.645)</td>
<td>(7.079)</td>
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<td>-8971795</td>
</tr>
<tr>
<td></td>
<td>(0.037)</td>
<td>(0.287)</td>
<td>(0.362)</td>
<td>(0.497)</td>
<td>(2.862)</td>
<td>(7.650)</td>
</tr>
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<td>PA-IS</td>
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<td>-3637889</td>
<td>2642948</td>
<td>-3530444</td>
<td>-24208272</td>
<td>-10806786</td>
</tr>
<tr>
<td>(ARM)</td>
<td>(0.013)</td>
<td>(2.582)</td>
<td>(3.997)</td>
<td>(3.753)</td>
<td>(1.665)</td>
<td>(2.807)</td>
</tr>
</tbody>
</table>

PV, Dur0, Dur1 and Dur2 respectively denote the present value and the gradients with respect to the three harmonic shocks, \( \Delta_0, \Delta_1 \) and \( \Delta_2 \). Vega1 and Vega2 denote the gradients with respect to the volatility factors \( \gamma \) and \( \eta \). Data in parentheses are percentage relative errors defined as standard deviations divided by absolute average values. Standard deviations are calculated from 100 sample runs.
estimators. Differentiating the price either analytically or approximately typically leads to an increase in the estimated variance. Importance sampling, even when the drift vector is optimally chosen with respect to price, clearly reduces the variance of the gradient estimators; the variance of the first-order gradient estimator in particular is reduced by nearly a half (Table 1).

The results for the second-order gradients with respect to the harmonic shocks, denoted Con0, Con1 and Con2, are presented in Table 2. The values in the lower triangular part of each estimator are the gradient estimates, while the values in the upper triangular part are the corresponding relative errors. The simulation errors tend to be larger compared to the first-order gradient estimates in all estimators. For the FD estimators, the errors accumulate as prices are differenced, and an increase in variance is an inevitable consequence. Our PA estimators, even though they rely on analytical gradients, also exhibit this variance increase. In retrospect this can be anticipated to some degree, since higher-order gradients tend to be far more sensitive to the parameters.

**TABLE 2** Hessian estimates with respect to harmonic shocks under the Hull–White two-factor interest rate model.

<table>
<thead>
<tr>
<th></th>
<th>Con1</th>
<th>Con2</th>
<th>Con3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PA-IS</td>
<td>(3.979)</td>
<td>(4.895)</td>
<td>(7.094)</td>
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<tr>
<td></td>
<td>529568396</td>
<td>(6.056)</td>
<td>(8.904)</td>
</tr>
<tr>
<td></td>
<td>−468925510</td>
<td>416547443</td>
<td>(13.441)</td>
</tr>
<tr>
<td></td>
<td>380548983</td>
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<td>FD-IS</td>
<td>(3.917)</td>
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<td>528532912</td>
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<td>(8.903)</td>
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<td>(13.246)</td>
</tr>
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<td></td>
<td>380389457</td>
<td>−330743048</td>
<td>266620391</td>
</tr>
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<td>(6.320)</td>
<td>(7.752)</td>
<td>(11.137)</td>
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<td></td>
<td>529059199</td>
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<td></td>
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<td>416637553</td>
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<tr>
<td></td>
<td>381009204</td>
<td>−336860741</td>
<td>269099424</td>
</tr>
<tr>
<td>FD</td>
<td>(7.406)</td>
<td>(9.193)</td>
<td>(12.861)</td>
</tr>
<tr>
<td></td>
<td>530920971</td>
<td>(11.460)</td>
<td>(16.255)</td>
</tr>
<tr>
<td></td>
<td>−463919105</td>
<td>406163018</td>
<td>(23.631)</td>
</tr>
<tr>
<td></td>
<td>382057267</td>
<td>−331886519</td>
<td>267141305</td>
</tr>
<tr>
<td>PA-IS</td>
<td>(12.863)</td>
<td>(11.503)</td>
<td>(13.779)</td>
</tr>
<tr>
<td>(ARM)</td>
<td>25974384</td>
<td>(28.322)</td>
<td>(1545.460)</td>
</tr>
<tr>
<td></td>
<td>−31187057</td>
<td>14416692</td>
<td>(8.714)</td>
</tr>
<tr>
<td></td>
<td>30058064</td>
<td>350084</td>
<td>−139819135</td>
</tr>
</tbody>
</table>

Con0, Con1 and Con2 respectively denote the second-order derivatives with respect to the harmonic shocks $\Delta_0$, $\Delta_1$ and $\Delta_2$. The values in the lower triangular part of each estimator are the estimates and the values in parentheses, in the upper triangular part, are percentage relative errors defined as standard deviations divided by absolute average values. Standard deviations are calculated from 100 sample runs.
The simulation errors across the estimators possess similar characteristics to those observed in the first-order gradient estimators, e.g., the simulation variances of the PA estimators are similar to but normally smaller than those of the FD estimators, and importance sampling leads to variance reduction of all estimators: the variances are reduced by almost a half.

We next observe the gradients with respect to the risk factors shown in Tables 3 and 4. The first observation is that the first-order gradients increase up to 20 years, slow over the long horizon, and drop significantly at maturity. This is an expected consequence of the tradeoff between increasing time horizon and decreasing cashflows. The overall behavior of the simulation errors is similar to

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Gradient estimates with respect to risk factors under the Hull–White two-factor model.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 m</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
</tr>
<tr>
<td>PA-IS</td>
<td>–2349</td>
</tr>
<tr>
<td>(ARM)</td>
<td>(0.002)</td>
</tr>
<tr>
<td></td>
<td>5y</td>
</tr>
<tr>
<td></td>
<td>(0.319)</td>
</tr>
<tr>
<td></td>
<td>(0.367)</td>
</tr>
<tr>
<td></td>
<td>(0.544)</td>
</tr>
<tr>
<td></td>
<td>(0.541)</td>
</tr>
</tbody>
</table>

Each column represents the gradients with respect to the spot rate (risk factor) at the corresponding month (year). Data in parentheses are percentage relative errors defined as standard deviations divided by absolute average values. Standard deviations are calculated from 100 sample runs.
### TABLE 4  Hessian estimates with respect to risk factors under the Hull–White two-factor model using FD method with importance sampling.

<table>
<thead>
<tr>
<th></th>
<th>1 m</th>
<th>3 m</th>
<th>6 m</th>
<th>1 y</th>
<th>2 y</th>
<th>3 y</th>
<th>4 y</th>
<th>5 y</th>
<th>7 y</th>
<th>9 y</th>
<th>10 y</th>
<th>15 y</th>
<th>20 y</th>
<th>30 y</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0.113)</td>
<td>(0.088)</td>
<td>(0.145)</td>
<td>(0.177)</td>
<td>(0.286)</td>
<td>(0.355)</td>
<td>(0.425)</td>
<td>(0.386)</td>
<td>(0.429)</td>
<td>(0.505)</td>
<td>(1.1786)</td>
<td>(0.890)</td>
<td>(1.174)</td>
<td>(0.958)</td>
</tr>
<tr>
<td>92</td>
<td>(0.150)</td>
<td>(0.188)</td>
<td>(0.401)</td>
<td>(0.475)</td>
<td>(0.496)</td>
<td>(0.585)</td>
<td>(0.580)</td>
<td>(0.596)</td>
<td>(0.617)</td>
<td>(1.908)</td>
<td>(0.984)</td>
<td>(1.293)</td>
<td>(1.040)</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>3548</td>
<td>(0.557)</td>
<td>(0.824)</td>
<td>(1.097)</td>
<td>(1.009)</td>
<td>(0.989)</td>
<td>(1.061)</td>
<td>(1.080)</td>
<td>(1.017)</td>
<td>(2.776)</td>
<td>(1.803)</td>
<td>(1.607)</td>
<td>(1.427)</td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>716</td>
<td>6835</td>
<td>200204</td>
<td>994961</td>
<td>(31.706)</td>
<td>(17.917)</td>
<td>(7.629)</td>
<td>(5.708)</td>
<td>(5.844)</td>
<td>(17.315)</td>
<td>(15.135)</td>
<td>(8.537)</td>
<td>(7.762)</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>1013</td>
<td>9677</td>
<td>76838</td>
<td>509024</td>
<td>1424121</td>
<td>(56.621)</td>
<td>(15.414)</td>
<td>(8.289)</td>
<td>(7.833)</td>
<td>(27.902)</td>
<td>(18.049)</td>
<td>(15.207)</td>
<td>(9.574)</td>
<td></td>
</tr>
<tr>
<td>167</td>
<td>2117</td>
<td>20158</td>
<td>155652</td>
<td>658589</td>
<td>1349889</td>
<td>2071500</td>
<td>10499770</td>
<td>(23.975)</td>
<td>(23.397)</td>
<td>(66.019)</td>
<td>(18.235)</td>
<td>(12.449)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>113</td>
<td>1439</td>
<td>13690</td>
<td>105440</td>
<td>442338</td>
<td>885104</td>
<td>1299555</td>
<td>2844279</td>
<td>6973127</td>
<td>11308336</td>
<td>(19.860)</td>
<td>(27.271)</td>
<td>(31.318)</td>
<td>(17.470)</td>
<td></td>
</tr>
<tr>
<td>109</td>
<td>12637</td>
<td>151438</td>
<td>1169731</td>
<td>4138696</td>
<td>5214132</td>
<td>2775911</td>
<td>1018188</td>
<td>417649</td>
<td>3802154</td>
<td>27584085</td>
<td>(72.509)</td>
<td>(65.231)</td>
<td>(31.019)</td>
<td></td>
</tr>
<tr>
<td>229</td>
<td>3598</td>
<td>47323</td>
<td>593671</td>
<td>3686806</td>
<td>9310376</td>
<td>15866413</td>
<td>27292619</td>
<td>16829775</td>
<td>16689805</td>
<td>33012126</td>
<td>(221.806)</td>
<td>(36.515)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>183</td>
<td>2314</td>
<td>21921</td>
<td>165888</td>
<td>679502</td>
<td>1297171</td>
<td>1645438</td>
<td>6692775</td>
<td>25671196</td>
<td>27237476</td>
<td>31630686</td>
<td>(13454087)</td>
<td>(1175911)</td>
<td>(185.507)</td>
<td></td>
</tr>
<tr>
<td>75.111</td>
<td>953</td>
<td>9082</td>
<td>69984</td>
<td>295289</td>
<td>591539</td>
<td>839648</td>
<td>1700237</td>
<td>2860349</td>
<td>2270190</td>
<td>9365193</td>
<td>22125135</td>
<td>24121366</td>
<td>7853561</td>
<td></td>
</tr>
</tbody>
</table>

The lower triangular part in the table is Hessian estimates and the upper triangular part with parentheses is percentage relative errors defined as standard deviations divided by absolute average values. Standard deviations are calculated from 100 sample runs.
those discussed earlier for the harmonic shocks. Although some of estimation errors of the second-order gradients are very high, the estimates are small in those cases and it should not affect the value-at-risk estimation in any meaningful way.

The first- and second-order gradients with respect to the spot rate risk factors essentially provide the basis for the key-rate durations and convexities, and the price sensitivity to any shape of yield curve movement can be described by these gradients. Thus, we can also derive the duration and convexity from the gradients with respect to risk factors instead of the harmonic shocks. For example, Dur0 and Con0 can be calculated from the following equations:

\[
\text{Dur}0 = \frac{\partial P}{\partial \Delta_0} = \frac{\partial P^T}{\partial R} \cdot \frac{\partial R}{\partial \Delta_0} = \frac{\partial P^T}{\partial R} \cdot 1
\]

\[
\text{Con}0 = \frac{\partial^2 P}{\partial \Delta_0^2} = \frac{\partial R^T}{\partial \Delta_0} \cdot \frac{\partial^2 P}{\partial R^2} \cdot \frac{\partial R}{\partial \Delta_0} = 1^T \cdot \frac{\partial^2 P}{\partial R^2} \cdot 1
\]

where \( R \) denotes the vector of risk factors and \( 1 \) denotes the vector of ones of appropriate dimension.

### 4.5.2 Value-at-risk

Both delta-normal and delta–gamma VAR can now be computed given the gradient estimates provided in Tables 3 and 4. VAR values at different confidence levels are presented in Table 5. In the table, the Monte Carlo VAR is obtained from 10,000 simulation runs.

**TABLE 5** VAR estimates by three different methods: delta-normal VAR, delta–gamma VAR and Monte Carlo VAR. Monte Carlo VARs are obtained from 10,000 runs.

<table>
<thead>
<tr>
<th>Confidence (%)</th>
<th>Delta-normal</th>
<th>Delta–gamma</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fixed-rate MBS</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>99.9</td>
<td>965373</td>
<td>869846</td>
<td>923892</td>
</tr>
<tr>
<td>99.0</td>
<td>749550</td>
<td>653874</td>
<td>651008</td>
</tr>
<tr>
<td>97.5</td>
<td>653304</td>
<td>551712</td>
<td>551608</td>
</tr>
<tr>
<td>95.0</td>
<td>571641</td>
<td>465517</td>
<td>468007</td>
</tr>
<tr>
<td>90.0</td>
<td>478312</td>
<td>365909</td>
<td>370739</td>
</tr>
<tr>
<td><strong>Adjustable-rate MBS</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>99.9</td>
<td>32403</td>
<td>174433</td>
<td>200166</td>
</tr>
<tr>
<td>99.0</td>
<td>25159</td>
<td>120576</td>
<td>130448</td>
</tr>
<tr>
<td>97.5</td>
<td>21928</td>
<td>99785</td>
<td>102201</td>
</tr>
<tr>
<td>95.0</td>
<td>19187</td>
<td>84280</td>
<td>79754</td>
</tr>
<tr>
<td>90.0</td>
<td>16055</td>
<td>68876</td>
<td>53565</td>
</tr>
</tbody>
</table>
In case of the fixed-rate MBS, both the delta-normal and delta–gamma VAR well approximate the Monte Carlo VAR; delta–gamma approximation is more accurate than delta-normal approximation in terms of distance from the Monte Carlo VAR. However, in the case of the adjustable-rate MBS, the delta-normal VAR very poorly approximates the Monte Carlo VAR, while the delta–gamma VAR still approximates it reasonably well. This can be traced to two reasons: the price sensitivities of the ARM with respect to the risk factors are much smaller than those of the FRM, as shown in Table 3, and so consequently is the VAR; the same error magnitude affects the VAR estimation of the ARM more significantly. Second, the caps and floors imposed on the coupon rate add a further element of non-linearity to the value of the ARM.

The VAR of the ARM also increases more rapidly with the confidence level compared to the FRM. The 99.9% VAR of the FRM is less than triple the 99.0% VAR, while the 99.9% VAR of the ARM is almost four times the 99.0% VAR. The 99.9% VAR of the ARM is only 21.7% of that of the FRM. Since the coupon rate of the ARM is adjusted annually with respect to the one-year Treasury rate, the interest rate risk is less severe compared to the FRM. The 99.9% VAR of the FRM is as large as 9.9% of its present value, and the 99.9% VAR of the ARM is as large as 1.9% of its present value.

4.5.3 Computational efficiency

The time required to compute the estimators are compared in Table 6. All simulations were performed on an IBM Thinkpad with a 1600 MHz Pentium M processor and 512 Mb RAM memory, and the programs were developed using Visual C++. For each row, the times (in seconds) are expressed cumulatively moving from left to right. As expected, our PA estimators require significantly shorter computation times than the corresponding FD estimators. Particularly noteworthy is that the computation times for the PA estimators are about the same for both the one-factor and two-factor interest rate models. In contrast, the computation times for the FD estimators approximately double as one moves from the one-factor to the two-factor model.

<table>
<thead>
<tr>
<th></th>
<th>IS</th>
<th>Price</th>
<th>1st order</th>
<th>2nd order</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>PA-IS</td>
<td>3.701</td>
<td>9.233</td>
<td>26.648</td>
<td>92.718</td>
<td>92.718</td>
</tr>
<tr>
<td>FD-IS</td>
<td>3.701</td>
<td>9.233</td>
<td>63.631</td>
<td>525.962</td>
<td>525.962</td>
</tr>
<tr>
<td>PA</td>
<td>2.103</td>
<td>19.237</td>
<td>88.341</td>
<td>88.341</td>
<td></td>
</tr>
<tr>
<td>FD</td>
<td>2.103</td>
<td>56.090</td>
<td>517.208</td>
<td>517.208</td>
<td></td>
</tr>
</tbody>
</table>

One simulation is completed by 1,000 iterations. Computation times are in seconds, and, from left to right, cumulative – i.e., the times in the fourth column (“1st order”) are times required to compute both the price (with or without importance sampling) and the first-order gradients. System specifications: IBM Thinkpad with 1600 MHz Pentium M processor and 512 Mb RAM memory. The programs are developed using Visual C++.
Noting that importance sampling is performed only at the first stage, the additional time required for importance sampling is not considerable relative to the total time. As noted earlier, the simulation errors will be much smaller when the importance sampling technique is used. Thus, one can expect that for the same degree of accuracy computation times will be much larger without importance sampling.

Computational efficiency is more critical for VAR estimation; pricing the MBS itself requires Monte Carlo simulation, so a simulation-based VAR estimation procedure requires an iteration of simulations. In our example, it takes about three hours to obtain the Monte Carlo VAR from 10,000 iterations, while it takes less than two minutes (mostly to compute gradients) to obtain the delta–gamma VAR. Once the gradient estimators are obtained, delta–gamma VAR can be estimated with only an incremental additional cost (a numerical calculation to solve Equation (16)).

5 CONCLUDING REMARKS

This paper has presented an efficient Monte Carlo simulation method for value-at-risk analysis (and, more fundamentally, pricing and sensitivity analysis) of mortgage-backed securities. Building on the preliminary work of Chen and Fu (2002), we extend their general methodology to estimate VAR for mortgage-backed securities under more computationally realistic scenarios. We also verify the effectiveness of importance sampling in our methodology. Because our PA methodology simultaneously computes a wide range of estimators in a single Monte Carlo simulation, whereas importance sampling is intended for a single estimator, there is no natural means of determining the optimal parameter value, $\mu$, required by importance sampling. Our simulation results show that choosing the $\mu$ that is optimal for pricing is sufficient to produce a significant reduction in errors in all of the estimates.

Another aim of our paper has been to assess the actual computational performance of the PA methodology for more realistic market-like situations. Our findings suggest that the PA-based methodology does lead to significant improvements in robustness and computational efficiency over their finite-difference-based counterparts. In a broader context, this paper demonstrates that even for the more complicated problem of MBS VAR estimation using two-factor interest rate models and complex prepayment models, the PA-based methodology is still a highly feasible one, in the sense of being both analytically tractable and computationally efficient. Our results also highlight the interplay between the interest rate and prepayment models, and emphasize the importance of choosing models that are both analytically tractable and consistent with each other.

Although the present paper focused exclusively on MBS, the methodology presented is in fact quite general, and can be applied to the VAR, pricing, and sensitivity analysis of any fixed-income security whose cashflows depend on a stochastic interest rate and also possess some degree of analytic tractability (eg, the availability of closed-form differentiable formulas for zero-coupon bond prices).
REFERENCES


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