Simulation-Based Actuator Selection for Redundantly Actuated Robot Mechanisms

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This article presents a simulation-based strategy for sizing the actuators of a redundantly actuated robotic mechanism. The class of robotic mechanisms we consider may contain one or more closed loops and possess an arbitrary number of active and passive joints, and the number of actuators may exceed the mechanism’s kinematic degrees of freedom. Our approach relies on a series of dynamic simulations of the mechanism, by applying Taguchi’s method to systematically perform the simulations. To efficiently perform each of the dynamic simulations, we develop, using tools from modern screw theory, new recursive algorithms for the forward and inverse dynamics of the class of redundantly actuated mechanisms described.

1. INTRODUCTION

An important aspect of robot design that, with few exceptions (for example, Thomas et al.1), is rarely addressed in the literature, is selecting the strengths of the joint actuators. Actuator sizing is an important factor in determining a robot’s dynamic performance and payload characteristics. In addition to the performance requirements imposed by the user (such as power consumption, desired end-effector velocity and acceleration characteristics, and maximum payload), actuator selection requires consideration of diverse factors such as the robot link masses and inertias, the presence of reduction gears, and joint characteristics.

While not directly focused on the actuator sizing problem, methods for relative actuator sizing based on a kineto-static analysis of the mechanism have been proposed in the literature by Kim,2 Kosuge,3 and Merlet.4,5 These approaches are an implicit acknowledgement of the difficulty of taking into account the dynamics in a systematic and tractable way; while computationally convenient, these methods do not address the most fundamental issues that determine the proper choice of actuators. Faced with these difficulties, current industrial practice for actuator selection relies mostly on a combination of rough

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kineto-static calculations and a designer’s experience and intuition. It is not surprising that such methods lead to excessively conservative choices for actuators that do not utilize their full capabilities, or worse, fail to meet the performance requirements. Thomas et al.\(^1\) was one of the first papers to recognize the shortcomings of existing methods for actuator selection, and presented a systematic approach for serial manipulators based on local dynamics criteria.

In this article, we present a global approach to actuator selection based on a series of dynamic simulations of the mechanism performed according to Taguchi’s method. Taguchi’s method was originally developed in the context of quality engineering, and offers a means of systematically performing experiments or simulations to optimize a given functionality criterion. For our problem we must select the actuators based only on a finite number of simulations. Rather than attempting to formulate an ad hoc statistical procedure, we instead show that actuator selection can be addressed entirely within the Taguchi framework, a framework that is quite familiar to engineers, and easy to learn and apply.

The most time-consuming step in the simulation-based actuator selection procedure is without doubt the dynamic simulation. A second contribution of this article is a set of efficient recursive algorithms for evaluating the dynamics of redundantly actuated closed chains. Redundantly actuated mechanisms are ubiquitous in biological systems, where, for example, muscles function as antagonistic actuators. Antagonistic and other redundant actuation schemes have recently been explored in the robotics literature, both as a means of improving dynamic performance and of eliminating kinematic singularities in the workspace by Matone and Roth.\(^6\) The dynamics of redundantly actuated mechanisms that contain no passive joints, such as cooperating manipulators, have been studied in the literature, as have the dynamics of exactly actuated closed chains. Redundantly actuated closed chains containing passive joints, however, do not appear to have been addressed in any detail.

Among various dynamics formulations for exactly actuated closed chains, Wittenburg\(^7\) introduces the notion of a reduced system, that is, a tree topology mechanism obtained by cutting the loops in a closed chain mechanism. This technique has been refined by Freeman,\(^8\) Lu and Zheng,\(^9\) and Wittenburg and Voltz.\(^10\) More recently, Murray and Lovell\(^11\) and Nakamura and Ghodoussi\(^12\) have independently applied D’Alembert’s Principle to derive the transformation \(\Phi\) between the external generalized forces of a closed chain and its corresponding reduced system. Such a transformation leads to, among other things, a set of equations of motion that does not require Lagrange multipliers. Moreover, existing recursive algorithms intended for open chains can also be applied to perform the heart of the closed chain dynamics computation. This approach is in fact equivalent to the classical method based on Lagrange multipliers, in which the Lagrange multipliers are now eliminated by coordinate partitioning.

Although in principle \(\Phi\) and \(\dot{\Phi}\) can be systematically evaluated for arbitrary closed chains, usually the complexity of the constraint equations makes this difficult for all but the simplest closed kinematic chains. Nakamura and Ghodoussi\(^12\) employ numerical techniques to approximately evaluate these quantities, but these also have the obvious disadvantages with respect to accuracy and computational efficiency. Taking the symbolic formulation for exactly actuated closed chains developed by Park et al.\(^13\) as our point of departure, in this article we extend these results to redundantly actuated closed chains. We derive exact symbolic formulations of \(\Phi\) and \(\dot{\Phi}\), together with analytic equations for both the inverse and forward dynamics, in a form that allows one to use the recursive algorithms derived for open chains.

The article is organized as follows. In Section 2 we present the dynamics of exactly actuated closed chains. In Section 3 we analytically derive the inverse and forward dynamics for redundantly actuated closed chains. Section 4 presents the framework for actuator sizing based on Taguchi’s method. Section 5 presents an application to the sizing of actuators for the Eclipse, a six degree-of-freedom (DOF), redundantly actuated, parallel mechanism designed for five-face rapid machining. A summary of our work is given in Section 6.

2. EXACTLY ACTUATED CLOSED CHAINS

We now briefly revisit the recursive formulation of an exactly actuated closed chain’s dynamics as given in Park et al.\(^13\) The paper supposes a certain degree of familiarity with Lie groups and the product of exponential formulas for open chains. The interested reader is referred to Brockett,\(^14\) Murray et al.,\(^15\) and Park et al.\(^16\) for details.

Recall that the dynamic equations for an open kinematic chain can be written as follows:

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau
\]  

where \(q\) denotes the joint position vector, \(M(q)\) is the
inertia matrix, and \( C \) and \( G \) represent the Coriolis/centrifugal and gravity terms, respectively. Before deriving the recursive algorithm for evaluating the dynamics, we begin with some notation and definitions.

Given a fixed frame, the position and orientation of a rigid body is described by an element of the Special Euclidean Group of rigid-body motions, denoted \( SE(3) \), given by \( 4 \times 4 \) matrices of the form:

\[
\begin{bmatrix}
R & p \\
0 & 1
\end{bmatrix}
\]

where \( R \in SO(3) \) is a \( 3 \times 3 \) rotation matrix that represents the orientation of the body-fixed frame, and \( p \in \mathbb{R}^3 \) represents the position of the origin of the same frame. Elements of \( SE(3) \) will be also be denoted by the pair \((R, p)\). \( SE(3) \) constitutes a group under matrix multiplication, and is an example of a Lie group.

The generalized velocity of a rigid body is described by an element of the Lie algebra of \( SE(3) \), denoted \( se(3) \), consisting of matrices of the form:

\[
\begin{bmatrix}
w & v \\
0 & 0
\end{bmatrix}
\]

where

\[
[w] = \begin{bmatrix}
0 & -w_3 & w_2 \\
w_3 & 0 & -w_1 \\
-w_2 & w_1 & 0
\end{bmatrix}
\]

and \( v \in \mathbb{R}^3 \). \( w \) and \( v \) can be used to represent the angular and linear velocities of the rigid body frame with respect to some reference frame. Elements of \( se(3) \) will also sometimes be denoted by the pair \((w, v)\).

Wrenches or generalized forces are described by elements of the dual space to \( se(3) \), denoted \( se(3)^* \). Recall that the dual space \( se(3)^* \) is the space of linear functions on \( se(3) \): if \( V = (w, v) \in se(3) \) and \( F = (m, f) \in se(3)^* \), then \( F : se(3) \to \mathbb{R} \) is defined by \( F(V) = F^TV = m^Tw + f^Tv \). Here \( m \) and \( f \) respectively correspond to moment and force. If \( X = (R, p) \) is an element of \( SE(3) \) and \( x = (w, v) \) an element of \( se(3) \), then the adjoint map \( Ad_X : se(3) \to se(3) \) admits the \( 6 \times 6 \) matrix representation:

\[
Ad_X(x) = \begin{bmatrix}
R & 0 \\
[p]R & R
\end{bmatrix}
\begin{bmatrix}
w \\
v
\end{bmatrix}
\]

where \([p] \) denotes the \( 3 \times 3 \) skew-symmetric matrix representation of \( p \in \mathbb{R}^3 \). Physically, the adjoint mapping describes how generalized velocities transform under a change of reference frame given by \( X = (R, p) \). It can be easily verified that \( Ad_X^{-1} = Ad_{X^{-1}} \) and \( Ad_X Ad_Y = Ad_{XY} \) for any \( X, Y \in SE(3) \). The dual operator \( Ad_X^* : se(3)^* \to se(3)^* \) admits the matrix representation given by the transpose of \( Ad_X \). That is, if \( z = (m, f) \in se(3)^* \), then:

\[
Ad_X^*(z) = \begin{bmatrix}
R^T & R^T[p]^T \\
0 & R^T
\end{bmatrix}
\begin{bmatrix}
m \\
f
\end{bmatrix}.
\]

Physically, the dual adjoint mapping describes how generalized forces transform under a change of reference frame given by \( X = (R, p) \).

Elements of Lie algebra can also be identified with a linear mapping between its Lie algebra via the Lie bracket. On matrix Lie algebras the Lie bracket is the matrix commutator: if \( A, B \in se(3) \), then \([A, B] = AB - BA \). Given \( x \in se(3) \), its adjoint representation is given by the linear map defined by \( ad_X^*(y) = [x, y] \). Physically, the mapping \( ad_X^*(y) \) and its adjoint \( ad_X^{**}(z) \) can be regarded as a generalization of the cross product operation to \( se(3) \) and \( se(3)^* \), respectively.

With the above background, we are now ready to describe the recursive dynamics algorithm for tree topology systems. Suppose the links and joints of the tree are numbered and increase outward from the base. Define the connectivity index \( \lambda(i) \) to be the number of the link that directly precedes link \( i \). Also define \( \mu(i) \) as the list of links that are direct successors of link \( i \). In this case, the recursive algorithm for open chain dynamics can be modified accordingly to describe the dynamics of tree systems. (See Park et al.\textsuperscript{13} for further details.)

- **Initialization**

\[
V_0, \dot{V}_0, F_{\mu_0}
\]

- **Forward recursion: for \( i = 1 \) to \( n \)**

\[
f_{\lambda(i), i} = M_i e^{S\theta_i} \\
V_i = Ad_{f_{i-1}} V_{\lambda(i)} + S_i \dot{q}_i \\
\dot{V}_i = S_i \dot{q}_i + Ad_{f_{i-1}}^\ast V_{\lambda(i)} + ad_{Ad_{f_{i-1}}^\ast V_{\lambda(i)}} (S_i \dot{q}_i)
\]

- **Backward recursion: for \( i = n \) to \( 1 \)**

\[
F_i = Ad_{f_{i+1}}^\ast F_{\mu(i)} + J_i \dot{V}_i - ad_{V_i}^\ast (J_i V_i) \\
\tau_i = S_i^\ast F_i
\]
where

- \( V_i \in \mathbb{R}^6 \) is the generalized velocity of link \( i \);
- \( V_0 = 0 \) (the base is assumed stationary) and \( V_0 = (0, -g) \), where \( g \in \mathbb{R}^3 \);
- \( f_{i-1} = M_i e^{5\theta} \) denotes the rigid body transformation from link \( i - 1 \) to \( i \) with \( M_i \in SE(3) \) and \( S_i \in se(3) \);
- \( F_i \in \mathbb{R}^6 \) is the wrench transmitted from link \( i - 1 \) to \( i \);
- \( J_i \) is the \( 6 \times 6 \) spatial inertia matrix of link \( i \). \( J_i \) is composed of the inertia matrix \( I_i \), mass \( m_i \), and the vector \( r_i \) from the link \( i \) frame to the center of mass of link \( i \) and has the following form:

\[
J_i = \begin{bmatrix}
I_i - m_i [r_i]_T & m_i [r_i]_T \\
- m_i [r_i]_T & m_i \cdot 1
\end{bmatrix}
\]

Here, 1 is the \( 3 \times 3 \) identity matrix.
- \( \tau_i \) is the applied torque at joint \( i \).

After expanding each of the above equations and following some matrix manipulation, it can be shown that the recursive algorithm for tree topology systems follows some matrix manipulation, it can be shown that the recursive algorithm for tree topology systems admits the following global matrix representation:

\[
V = G_1 S q + G_1 P_{0,1} V_0
\]

\[
\dot{V} = G_1 S \ddot{q} + G_1 \dot{a} S \dot{q} + G_1 \alpha S \gamma V + G_1 \dot{a} \dot{S} P_{0,1} V_0
\]

\[
F = G_1^T J \dot{V} + G_1^T \dot{a} \dot{V} (J \dot{V}) + G_1^T P_{1,1} F_{1,U}
\]

\[
\tau = S^T F
\]

Combining these four equations, the equations of motion for a tree system can be expressed as:

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + \varphi(q) + J_i^T (q) F_{1,U} = \tau
\]

with

\[
M(q) = S^T G_1^T J G_1 S
\]

\[
C(q, \dot{q}) = S^T G_1^T (J G_1 \dot{a} S \dot{q} + \dot{a} \dot{q}) G_1 S
\]

\[
\varphi(q) = S^T G_1^T f G_1 \dot{P}_{0,1} \dot{V}_0
\]

\[
J_i(q) = P_{1,1} G_1 S
\]

where

- \( V = \text{column}[V_1, \ldots, V_n] \in \mathbb{R}^{6n} \);
- \( V_0 = \text{column}[V_{0(1)}, \ldots, V_{0(n)}] \in \mathbb{R}^{6n} \);
- \( S = \text{diag}[S_1, \ldots, S_n] \in \mathbb{R}^{6n \times n} \);
- \( J = \text{diag}[J_1, \ldots, J_n] \in \mathbb{R}^{6n \times 6n} \);
- \( F_{1,U} = \text{column}[F_{1,i}, \ldots, F_{1,n}] \);
- \( \alpha_{\lambda j} = \text{diag}[-\alpha_{\lambda j}, \ldots, -\alpha_{\lambda j} \alpha_{\lambda j}] \in \mathbb{R}^{6n \times 6n} \);
- \( \alpha_{\lambda i} = \text{diag}[-\alpha_{\lambda i}, \ldots, -\alpha_{\lambda i}] \);
- \( \Gamma_{\lambda} = M_{\lambda} \Theta_{\lambda} \in \mathbb{R}^{6n \times 6n} \);
- \( \Theta_{\lambda}(i, j) = \begin{cases} 1 & \text{if } \lambda(i) = j \\ 0 & \text{otherwise} \end{cases} \)
- \( P_{1,1}^T = \begin{pmatrix} P_{1,1}^T (1, 1) & \cdots & P_{1,1}^T (1, n) \\ \vdots & \ddots & \vdots \\ P_{1,1}^T (n, 1) & \cdots & P_{1,1}^T (n, n) \end{pmatrix} \);
- \( P_{0,\lambda} = M_{\lambda} B \);
- \( B = \text{column}[B_1, \ldots, B_n] \in \mathbb{R}^{6n \times 6n} \);
- \( B_i = \begin{cases} 1 & \text{if } \lambda(i) = 0 \\ 0 & \text{otherwise} \end{cases} \)
- \( G_1 = \begin{pmatrix} I & 0 & \cdots & 0 \\ G_1 (2, 1) & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_1 (n, 1) & G_1 (n, 2) & \cdots & I \end{pmatrix} \);
- \( G_1(i, j) = \begin{cases} Ad_{\lambda i}^{-1} & \text{if link } j \text{ precedes link } i \\ 0 & \text{otherwise} \end{cases} \)
- \( f_{j,i} = f_{\lambda^{-1}(i)}(i, \lambda^{-1}(i), \lambda^{-2}(i), \ldots, f_{1,0}, f_{0,i}) \) for \( i > j \).

The dynamics of exactly actuated closed chains can be obtained from the above by applying the D’Alembert Principle. Without any loss of generality, assume that the closed chain consists of a total of \( k \) one DOF joints, \( n \) of which are actuated. Since the chain is exactly actuated, the number of actuated joints is equal to the kinematic degrees of freedom of the mechanism, so that the number of passive joints is \( n - k \). We first convert this chain to a reduced system by selectively detaching certain joints such that the overall reduced mechanism undertakes a tree topology structure. The two systems (the original and the reduced one) are assumed to experience the same external forces and undergo the same motion.

Let the independent generalized coordinates of the original and reduced systems be given by \( \mathbf{q}_a \in \mathbb{R}^n \) and \( \mathbf{q}_r \in \mathbb{R}^m \), respectively, where \( n < m \). Also, let \( \mathbf{r}_a \in \mathbb{R}^n \) and \( \mathbf{r}_r \in \mathbb{R}^m \) represent the generalized actuator forces for the original and reduced systems, respectively. Then according to D’Alembert’s Principle, the virtual work performed by the generalized actuator forces for both systems must be equal:

\[
\delta \mathbf{q}_a^T \mathbf{r}_a = \delta \mathbf{q}_r^T \mathbf{r}_r
\]

Because the reduced system presumably undergoes only motions consistent with the constraints of the original closed chain mechanism, \( \mathbf{q}_r \) is functionally dependent on \( \mathbf{q}_a \), i.e., \( \mathbf{q}_r = \mathbf{q}_a (\mathbf{q}_a) \). This dependence
can also be expressed in differential form as:

\[ \delta \dot{q}_r = \Phi \delta \ddot{q}_a, \quad \Phi = \frac{\delta \dot{q}_r}{\delta \ddot{q}_a} \]  \hspace{1cm} (12)

By substituting this into Eq. (11), the generalized actuating force transformation for the two systems appears as:

\[ \tau_a = \Phi^T \tau_r \]  \hspace{1cm} (13)

which is simply a position-dependent linear transformation.

The dynamics of the closed chain reduces to the dynamics of a tree topology system of the form:

\[ \tau_r = M_r(q_r)\ddot{q}_r + C_r(q_r, \dot{q}_r)\dot{q}_r + \psi_r(q_r) + J^T_r(q_r)F_{t,U} \]  \hspace{1cm} (14)

If we now apply the generalized actuating force transformation of Eq. (13) to the reduced system, we obtain:

\[ \tau_a = \Phi^T [M_r(q_r)\ddot{q}_r + C_r(q_r, \dot{q}_r)\dot{q}_r + \psi_r(q_r) + J^T_r(q_r)F_{t,U}] \]  \hspace{1cm} (15)

where \( \ddot{q}_r \) can be obtained by differentiating the constraint Eq. (12), that is:

\[ \ddot{q}_r = \Phi \ddot{q}_a + \Phi \dot{q}_a \]  \hspace{1cm} (16)

Inserting this into Eq. (15), we get the closed chain equations:

\[ \tau_a = M_a(q_a)\ddot{q}_a + C_a(q_a, \dot{q}_a)\dot{q}_a + N_a(q_a) \]  \hspace{1cm} (17)

where

\begin{align*}
M_a(q_a) &= \Phi^T M_r(q_r(q_a)) \Phi \\
C_a(q_a, \dot{q}_a) &= (\Phi^T M_r(q_r(q_a))) \Phi + \Phi^T C_r(q_r(q_a)) \Phi \dot{q}_a \\
N_a(q_a) &= \Phi^T (\psi_r(q_r(q_a)) + J^T_r(q_r(q_a))F_{t,U})
\end{align*}  \hspace{1cm} (18-20)

with \( M_r(q_r(q_a)), C_r(q_r(q_a)), \psi_r(q_r(q_a)) \) and \( J_r(q_r(q_a)) \) given by Eqs. (7), (8), (9), and (10), respectively.

Essential to the above dynamic formulation is the evaluation of the matrices \( \Phi \) and \( \Phi \). Typical approaches appeal to numerical methods for the generalized force transformation and its derivative. Even though in general it is not possible to express the dependence \( \psi_r(q_r(q_a)) \) analytically, it is still possible to obtain a closed-form analytic expression for \( \Phi \) and \( \Phi \) by appropriately differentiating the kinematic constraint equations of the closed chain. Since the major difference between the dynamics of exactly actuated closed chains and redundantly actuated ones resides in how to evaluate the generalized force transformation of the two systems, this will be presented in the next section for both the exactly and redundantly actuated cases.

### 3. Dynamics of Redundantly Actuated Closed Chains

Following essentially the same procedure as outlined in the previous section, given a redundantly actuated, multiple-loop, closed chain (that is, the number of actuators exceeds the number of kinematic degrees of freedom of the mechanism), we first convert it to a reduced system by selectively detaching some of the passive joints, or cutting the loops so that the resulting mechanism assumes a tree topology structure and undergoes the same motion as the original system. We then solve the dynamics of the corresponding reduced system, and convert the solution of the reduced system into that of the original closed chain by applying a suitable transformation.

#### 3.1. Inverse Dynamics

We first consider the inverse dynamics; that is, given the desired end-effector trajectory, find a set of actuator torques that generates this motion. Assume that the closed chain mechanism consists of a total of \( k \) joints, \( n \) of which are actuated. Suppose there are \( c \) kinematic loops, which can be expressed as:

\[ h_i(q) = I, \quad i = 1, \ldots, c \]  \hspace{1cm} (21)

where \( h_i: \mathbb{R}^k \rightarrow SE(3) \) is the \( i \)th constraint equation, and \( I \) is the \( 4 \times 4 \) identity matrix. These constraints also apply for the corresponding reduced system because of the assumption that the reduced system undergoes motions consistent with the constraints imposed on the original system only.

Setting \( h_i h_i^{-1} = 0, i = 1, \ldots, c \) leads to a set of \( 6c \) equations, where each \( h_i h_i^{-1} \) is an element of \( se(3) \) that now corresponds to the generalized velocity in the fixed (or space) frame. (See Park et al.\(^{16}\) and Ploen and Park.\(^{17}\) The \( 6c \) equations can be written as:

\[ J_a^i \dot{q}_a + J_p^i \dot{q}_p = 0, \quad i = 1, \ldots, c \]  \hspace{1cm} (22)

where \( q_a \in \mathbb{R}^n \) is the vector of active (or independent) joints, \( q_p \in \mathbb{R}^{k-n} \) is the vector of passive (or dependent) joints, \( J^i \) is the \( 6 \times k \) spatial Jacobian corresponding to
the map \( h_1, J_1 \) is a submatrix of \( J^1 \) constructed by eliminating columns corresponding to the passive joints while keeping only the active joint columns, and \( J_p^1 \) is a submatrix of \( J^1 \), constructed by eliminating columns corresponding to the active joints while keeping only the passive joint columns.

If we further define:

\[
J_A = \text{column}[J_{a_1}^1, J_{a_2}^1, \ldots, J_{a_n}^1] \in \mathbb{R}^{6c \times n}
\]

\[
J_p = \text{column}[J_{p_1}^1, J_{p_2}^1, \ldots, J_{p_k}^1] \in \mathbb{R}^{6c \times (k-n)}
\]

then (22) can be written as:

\[
J_A \dot{q}_a + J_p \dot{q}_p = 0
\]

If the mechanism were exactly actuated (that is, \( k - 6c = n \)), and the constraints were chosen to be independent, \( J_p \) would be a square matrix. If, moreover, the closed chain is not in a singular configuration, then \( J_p \) will be invertible, and:

\[
\dot{q}_p = -J_p^{-1} J_A \dot{q}_a
\]

from which it follows that:

\[
\frac{\partial \dot{q}_p}{\partial \dot{q}_a} = -J_p^{-1} J_A
\]

Continuing with the exactly actuated case, let \( q_a \) and \( q_r \) denote a set of independent, generalized coordinates for the closed chain and its corresponding reduced system, respectively. In this situation, \( q_r \) consists of the actuated joints \( q_a \) plus those components of the passive joints \( q_p \) that have not been removed during the reduction stage (that is, when the chain is converted to a reduced system). Without loss of generality, we reorder the components of \( q_p \) as \( \dot{q}_p = (q_{pr}, \dot{q}_{pw}) \), where \( \dot{q}_{pr} \) corresponds to the passive joints that are also part of the reduced system, and \( \dot{q}_{pw} \) corresponds to the passive joints that have been removed during the reduction process. We also assume, without loss of generality, that the components of \( q_r \) are ordered as \( q_r = (q_{ar}, q_{pr}) \), where \( q_{ar} \) is identical to \( q_a \). We also define the selection matrix \( S \) as a constant diagonal matrix with components 1 or 0, such that \( q_{pr} = S \dot{q}_p \). Then the general force transformation will be of the form:

\[
\Phi_{\text{exact}} = \frac{\partial q_r}{\partial \dot{q}_a} = \begin{bmatrix} I \\ -SJ_p^{-1} J_A \end{bmatrix}
\]

From D’Alembert’s Principle, the virtual work performed by the two systems must be equal, so that (11) still applies.

\[
\tau_a^T \delta q_a = \tau_r^T \delta q_r
\]

Note that other forces acting on the reduced system, such as forces applied at the end-effector, can also be included in the above equation, via the transpose of the appropriate Jacobian.

By substituting Eq. (28) into Eq. (11), we obtain:

\[
\tau_a^T \delta q_a = \tau_r^T \Phi_{\text{exact}} \delta q_a
\]

Since this is valid for any virtual displacement \( \delta q_a \), we have:

\[
\tau_a = \Phi_{\text{exact}}^T \tau_r = \tau_{ar} - (J_A^T (J_p^T)^{-1})^T \tau_{pr}
\]

Now consider the redundantly actuated case (that is, \( k - 6c < n \)). Although \( J_p \) will then be overdetermined, as long as the physical constraints imposed by the mechanism are satisfied, a unique solution for \( \dot{q}_p \) can still be obtained as:

\[
\dot{q}_p = -(J_p^T J_p)^{-1} J_p^T J_A \dot{q}_a = -J_p J_A \dot{q}_a
\]

from which it follows that:

\[
\frac{\partial \dot{q}_p}{\partial \dot{q}_a} = -J_p J_A
\]

The generalized force transformation will therefore be of the form:

\[
\Phi_{\text{redund}} = \frac{\partial q_r}{\partial \dot{q}_a} = \begin{bmatrix} I \\ -SJ_p J_A \end{bmatrix}
\]

Once again appealing to D’Alembert’s Principle, by substituting Eq. (33) into Eq. (11), we obtain:

\[
\tau_a = \Phi_{\text{redund}}^T \tau_r = \tau_{ar} - (J_A^T (J_p^T)^{-1})^T \tau_{pr}
\]

Since the null space of \( J_p^T \) is not trivial, the general solution can be obtained by adding a null space term, as follows.

\[
\tau_a = \tau_{ar} - J_A \left\{ J_p (J_p^T J_p)^{-1} S^T \tau_{pr} + \sum_{i=1}^{n_p} \beta_i N_i \right\}
\]

\[
= [I - J_A J_p (J_p^T J_p)^{-1} S^T] \tau_{ar} - J_A^T N (J_p^T) \beta^T
\]
where \( \{N_1, N_2, \ldots, N_n\} \) is a basis for the null space of \( J_p^T \), which we simply denote as \( \mathcal{N}(J_p) \), \( \beta_i \in \mathbb{R}, i = 1, \ldots, n_r \), are scalar constants, and \( n_r = 6c - (k - n) \) represents the number of redundant actuators. The above can be verified by noting that from (37),

\[
\delta q_a^T \tau_a = \delta q_a^T \tau_{ar} - \delta q_a^T J_A^T (J_p^T)^{-1} S^T \tau_{pr} - \delta q_a^T J_A^T \sum_{i=1}^{n_r} \beta_i N_i
\]  

(38)

Also from (25):

\[
\delta q_a^T J_A = -\delta q_p^T J_p^T
\]  

(39)

Therefore,

\[
\delta q_a^T \tau_a = \delta q_a^T \tau_{ar} + (S\delta q_p)^T \tau_{pr} + \delta q_p^T \sum_{i=1}^{n_r} \beta_i (J_p^T N_i)
\]  

(40)

Since \( N_i, i = 1, \ldots, n_r \) are elements of \( \mathcal{N}(J_p^T) \), the third term on the right side is zero. Moreover, \( q_a = q_{ar} \) and \( S \delta q_p = q_{pr} \), which verifies the claim.

In summary, given the motion of the end-effector, and applied external forces, the inverse dynamics of a redundantly actuated closed chain can be evaluated as follows:

- Compute the velocity and acceleration of the end-effector.
- Solve the inverse kinematics of the original mechanism and compute the joint velocities and accelerations (singularity check included).
- Solve the inverse dynamics of the reduced system and convert the torque of the reduced system to the original system.

Based on the above inverse dynamics algorithm, we now derive an expression for minimum torque distribution. Eq. (37) can be written as:

\[
\tau = Ar_r + b\beta
\]  

(41)

where \( \beta = (\beta_1, \ldots, \beta_n)^T, A = \Phi_{\text{redu}}^T, \) and \( b = -J_A^T N(J_p) \). Since the inverse dynamics admits more than one solution, we look for a minimum norm solution on the vector of actuator torques. For this, the coefficients \( \beta \) are simply given as:

\[
\beta = -b^+ Ar_r
\]  

(42)

where \( b^+ \) is the left pseudo inverse matrix for \( \mathcal{N}(J_p^T) \), parameters that determine the minimum torque distribution of the closed chain actuators. This also corresponds to the case in which the actuators have the same power. If an additional diagonal weighting matrix \( W \) is appropriately chosen, with elements on the diagonal proportional to the size of the actuators, Eq. (41) then becomes:

\[
W \tau = W Ar_r + Wb\beta
\]  

(43)

and the optimal coefficients \( \beta \) corresponding to the minimum torque solution are:

\[
\beta = -(Wb)^+ W Ar_r
\]  

(44)

3.2. Forward Dynamics

Since the exactly and the redundantly actuated systems both have the same reduced system, the following hold:

\[
J_A^T \delta q_A^T = J_p^T \delta q_p^T = 0
\]  

(45)

\[
J_A^T \delta q_{A\text{redu}}^T = J_p^T \delta q_{pr}^T = 0
\]  

(46)

\[
\tau_{a\text{exact}} = \Phi_A^T \tau_r
\]  

(47)

\[
\tau_{a\text{redu}} = \Phi_{A\text{redu}}^T \tau_r
\]  

(48)

From the last two equations we have:

\[
\tau_{a\text{redu}} = (\Phi_{A\text{redu}}^T)(\Phi_A^T)^{-1} \tau_{a\text{exact}}
\]  

(49)

Therefore, given the applied actuator forces and torques, the forward dynamics of the redundantly actuated system can be computed according to the following steps:

- Solve the reduced system forward dynamics of the exactly actuated system. Recursive algorithms for the forward dynamics of open chain and tree topology systems can be used for this step.\(^{17,18}\)
- Convert the results to the redundantly actuated case via Eq. (49).

4. TAGUCHI’S METHOD FOR SIMULATION-BASED ACTUATOR SIZING

The Taguchi method was originally developed in the context of quality engineering, and refers to a methodology for the evaluation and improvement of a product’s robustness, tolerance specifications, quality management for the production process, and evaluating the losses associated with a product’s functional...
Part of the success and appeal of the Taguchi method is that it is general enough to be applied to a wide range of problems that seemingly have very little to do with quality engineering. It also does not draw upon a complicated probability or statistical analysis, which is particularly relevant for problems in which prior distributions may not be available or reliable. In this section we apply the Taguchi method in designing the experiments (or in this case the dynamic simulations) for actuator sizing, and analyzing the results to determine the optimal actuator sizes.

Under the Taguchi framework, the basic steps for designing an experiment or simulation are as follows:

1. Establish the objectives: A performance criterion that reflects how well the system is performing needs to be established. For our task the objectives are to determine the optimal actuator sizes, considering the dynamics of the redundantly actuated system, manufacturing requirements, and the preference for minimum torque motions. The signal-to-noise (S/N ratio), denoted \( \eta \), reflects the ability of the system to perform well in the presence of noise, where noise here refers to any sources of variation in the objective function, for example, manufacturing imperfections, environmental effects, and wear. The Taguchi method classifies the S/N ratio into three types: nominal-is-best, smaller-is-better, and larger-is-better. Table I shows typical S/N ratios for each classification type; here \( y_i \) denotes the results of the \( i \)-th run of a set of \( n \) simulated or experimental trials.

For each of these cases, we seek to find the combination of controllable factors that maximizes the S/N ratio. The S/N ratio can be viewed as a measure of functional robustness. By solving the inverse dynamics of the redundantly actuated closed chain mechanism for a given end-effector trajectory and controllable and noise factors at different levels, the time history of the joint torques \( \tau_i \) can be obtained. From this, the sums of the torque variations \( \Delta \tau_i \) are determined (these are exactly the measurable characteristics \( y_i \)) and the S/N ratios can be computed. Because we are generally interested in minimizing the required torques, the smaller-is-better S/N ratio formula will be used.

2. Selecting the independent variables: The independent (controllable) and the dependent (or noise) variables need to be identified and separated. For our actuator sizing problem, the controllable factors are given by the actuator sizes. Since we have a means to optimally balance the size of the actuators for power saving, that is, the diagonal weighting matrix \( W \) of (44), the controllable factors are the ratios \( M_i/M_1 \), where the \( M_i \) represents the power of the mechanism’s actuators. Hence the elements of the diagonal matrix \( W \) are obtained from the above ratios. The noise factors are the velocity, acceleration, and external forces applied at the end-effector. For both the controllable and noise factors, appropriate levels need to be determined for experimentation.

3. Selecting an orthogonal array: An exhaustive experimentation of all possibilities requires the testing of all combinations of the factor levels under study. For example, a study involving 13 factors at three levels each would require 1,594,323 experiments. Orthogonal arrays produce smaller, less costly experiments that have high rates of reproducibility. By using a \( L_{27}(3^{13}) \) orthogonal array, one can conduct a study involving 13 factors at three levels with only 27 experiments. We use an experimental setup that requires an orthogonal design for both controllable and noise factors; the complete experiment is simply the product of these two orthogonal designs.

4. Analysis: The average value of the S/N ratio for each level must be computed and plotted. From this, a relative comparison of the plotted point slopes can be performed, and the factors with the least and greatest effect on the S/N ratio can be identified. Since in any signal-to-noise analysis the greatest S/N ratio is preferred, for strong effects the highest points indicate the preferred level.

Table I. Typical S/N ratios.

<table>
<thead>
<tr>
<th>Objective</th>
<th>S/N ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal-is-best</td>
<td>( S/N = 10 \log \frac{\eta^2}{\sigma^2} )</td>
</tr>
<tr>
<td>Smaller-is-better</td>
<td>( S/N = -10 \log \left( \frac{1}{n} \sum_{i=1}^{n} y_i^2 \right) )</td>
</tr>
<tr>
<td>Larger-is-better</td>
<td>( S/N = -10 \log \left( \frac{1}{n} \sum_{i=1}^{n} \frac{y_i^2}{\sigma^2} \right) )</td>
</tr>
</tbody>
</table>

Developed from a biological perspective, the genetic algorithms have been demonstrated to have an
intimate connection to generally traditional statistical methods, and in particular to the design of experiments branch. Some of these differences, in the specific context of actuator sizing, are as follows: (1) The design of experiments used in the Taguchi method relies on an explicit model (the inverse dynamics) that accounts for the size of the actuators, while genetic algorithms do not explicitly use any modeling concepts; (2) The inverse dynamics simulation used in Taguchi’s method incorporates cumulative information obtained during the simulation process for some representative machining paths, whereas genetic algorithms do not remember any results of previous trials; (3) A certain degree of human interfacing and interpretation is required in the selection of both independent variables and noise factors, and representative machining trajectories. Genetic algorithms can proceed automatically from a random subset of the whole world (the machining trajectories in this case), although the use of a carefully structured orthogonal array; that is, using as few trials as possible can reduce substantially the computational time and offer better results (particularly in small problems such as the actuator sizing one).

That is why, in the context of a dynamic simulation-based actuator sizing problem, a statistical framework such as the Taguchi method matches nicely the designer’s judgment and provides considerable and suitable insight in the matter. In the next section, we provide a detailed case study that illustrates the Taguchi-based actuator sizing methodology for a novel parallel mechanism.

5. CASE STUDY: THE ECLIPSE

5.1. Description of the Eclipse

The Eclipse\textsuperscript{21,22} is a novel six DOF parallel mechanism designed for rapid machining. As shown in Figures 1 and 2, the Eclipse mechanism consists of three serial chains that move independently on a circular track. Each chain has a CPRS structure, where C denotes a circular prismatic joint, P a linear prismatic joint, R a revolute joint, and S a spherical (ball) joint. The three circular prismatic joints, three prismatic joints, and one of the revolute joints are actuated. The unique structure of the Eclipse gives it a singularity free workspace that is particularly suited for rapid machining; the spindle is able to continuously and smoothly machine along any of the five open faces of a cube, and to continuously machine across contiguous faces.

Figure 1. Kinematic structure of the Eclipse: Perspective view.

Our simulations were performed by using the Eclipse prototype shown in Figure 3. The mechanism has a cylindrical workspace of radius 100 mm and height 100 mm. The overall dimensions of the prototype are $1,590 \text{ mm} \times 1,340 \text{ mm} \times 1,820.5 \text{ mm}$, and the maximum feed rate of the tool tip is 1.5 m/min. The vertical columns slide independently on a pair of circular guide bearings of diameters of 500 and 600 mm, respectively. Each of the vertical columns is driven on the circular guide by a ring gear and a pinion system coupled directly to a servomotor. On each column there is a carriage that moves vertically along the linear guide of the column. This carriage motion is

Figure 2. Kinematic structure of the Eclipse: Top and side views.

\[ R_c \]
\[ B, C_i = L_i, i = 1, 2, 3 \]
\[ B_1, B_2, B_3 = D_1, B, B_2 = D_1 \]
realized by a servomotor and a ball screw transmission. Through a pin joint a fixed rod is attached to each carriage. The other end of the fixed rod is linked to the spindle platform through a ball and socket joint. To avoid internal workspace singularities, an additional actuator is installed at the pin joint on the lower carriage.\(^{22}\)

5.2. Simulation Results

Following the procedure described earlier, we now present the simulation results for sizing the actuators of the Eclipse. Initially, six controllable factors were chosen, that is, \(M_i/M_1, i = 2, \ldots, 7\), where \(M_1\) corresponds to the actuator located at joint \(A_1\), while \(M_2\) through \(M_7\) correspond to the actuators located at joints \(C_1\) prismatic, \(C_1\) revolute, \(A_2\) prismatic, \(A_3\), and \(C_3\) prismatic, respectively; see Figure 1. However, due to the similar structure of the last two chains, the controllable factors \(M_4/M_1, M_5/M_1\) and \(M_6/M_1, M_7/M_1\) have the same values, so that only the first two of these are considered in the analysis. The levels of the controllable factors are presented in Table II.

We chose the velocity and acceleration of the end-effector as the noise factors, levels of which are presented in Table III. For the orthogonal array, we use an \(L_9(3^4)\) orthogonal array, for which nine experiments at each level are sufficient. (See Table IV.)

![Figure 3. The prototype Eclipse.](image-url)
The test path for the inverse dynamics is selected to represent a typical motion of the Eclipse, that is, linear motion along the x and y axes, and tilting about the z axis. For this, the original mechanism is first transformed into an equivalent one by considering the three base circular joints as connecting to the center of the fixed guide. The equivalent mechanism is finally altered into a reduced system by virtually detaching one of the base joints. The inverse dynamics of this system is then solved according to the formulation presented in Section 3, that is, Eqs. (21), (32) to (38) and (41) to (44). The elements of the diagonal weighting matrix $W$ from Eq. (44) are given by the chosen controllable factors and finally the vector of actuator torques (time histories) is obtained as a minimum norm solution. The algorithm is written in MatLab version 5.2 and each inverse dynamics simulation run took about one minute on a Pentium 266 MHz PC. For each experiment at three noise levels, the sum of the joint torque variations (obtained from the inverse dynamics) is calculated, and the S/N ratios are obtained as:

$$\eta_i = -10 \log \left( \frac{1}{3} \sum_{i=1}^{3} y_{ij}^2 \right)$$

(50)

where the $y_{ij}$ are the torque variations for experiment $i = 1, \ldots, 9$ and noise factor level $j = 1, 2, 3$.

Having determined the S/N ratios, the average response for each level is plotted in Figure 4, where $A_j$, $B_j$, $C_j$, and $D_j$ correspond to the average S/N ratios for the four controllable factors $M_2/M_1$, $M_3/M_1$, $M_4/M_1$, and $M_5/M_1$. Figure 4 indicates that the first three controllable factors, $M_2/M_1$, $M_3/M_1$, and $M_4/M_1$, strongly affect actuator sizing, while the fourth factor, $M_5/M_1$, has less influence. Accordingly, the size of the actuators is given by Level III for $M_2/M_1$, $M_4/M_1$, and $M_5/M_1$, and Level I for $M_3/M_1$. (See Table V.)

When using the Taguchi method, a confirmation run at the preferred levels is vital for checking the reproducibility of the results and for confirming the assumptions made in the path planning and design of experiments. In our case, the recommended settings were used in a new experiment. (See the last row of Table IV.) The results in this case showed that the obtained S/N ratios agree with the plots of Figure 4, which confirms the optimal levels obtained for actuator sizing.

**Table V.** Optimal actuator sizing.

<table>
<thead>
<tr>
<th>Actuator</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$M_5$</th>
<th>$M_6$</th>
<th>$M_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comparative sizing</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>1.5</td>
<td>1</td>
<td>1.5</td>
<td>1</td>
</tr>
</tbody>
</table>
6. CONCLUSION

By using a combination of the Taguchi method of quality engineering and methods from Lie groups for the dynamics equations of redundantly actuated parallel mechanisms, we have presented a dynamic simulation-based method for actuator sizing. The actuator selection problem can be framed entirely within the Taguchi method, without relying on ad hoc statistical procedures that may be difficult for engineers to grasp. The actuators are selected based on a finite number of dynamic simulations that make use of an efficient set of recursive algorithms for the dynamics of redundantly actuated closed chains. Interactions among the controllable variables (actuator sizes) are considered through the dynamics equations for specific noise factors (manufacturing requirements).

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REFERENCES