Design, Analysis and Control of a Wheeled Mobile Robot with a Nonholonomic Spherical CVT
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Abstract

This article reports on the design, analysis and control of a new type of wheeled mobile robot based on a nonholonomic spherical continuously variable transmission (S-CVT). Our S-CVT based mobile robot is designed to increase the run time (i.e., the length of time in which the robot can be operated), and to achieve full planar accessibility with the design of a novel pivoting device that takes advantage of the flexibility of the S-CVT. We examine the sources of power loss in the S-CVT, in particular spin loss. For a quantitative analysis of spin loss of the S-CVT, we develop a friction model for the S-CVT, and perform an in-depth contact analysis based on the relative velocity field and normal pressure distribution. We also present a nonlinear shifting controller based on feedback linearization that takes into account the dynamics of the S-CVT. To evaluate the energy efficiency of our mobile robot and the performance of the S-CVT as a machine element, we perform experiments with a hardware prototype. The results are benchmarked numerically with a differential drive type mobile robot equipped with a reduction gear.

KEY WORDS—continuously variable transmission, spin loss, feedback linearization, mobile robot

1. Introduction

1.1. Continuously Variable Transmissions and Nonholonomy

Depending on one’s perspective, the feature of nonholonomy in input-output systems, which can be characterized by a reachable space whose dimension is larger than that of its input space, can be both a curse and an advantage. Although nonholonomy enables one to maneuver freely in an n-dimensional state space with fewer than n actuators, the associated mathematical complexity makes control and planning of such systems substantially more challenging. Examples of robotic systems with nonholonomy abound in the literature, e.g., systems with rolling constraints like a wheeled mobile robot, a car pulling multiple trailers, multibody systems subject to angular momentum conservation laws like a free-flying space robot, or simply a ball rolling on a plane. In most of these systems nonholonomy occurs in a natural way rather than by any intentional design.

More recently, robots that deliberately exploit nonholonomy to advantage have been developed. The Cobot (Moore et al. 1999) is a physically passive robotic device that provides virtual gliding surfaces for redirecting human-powered motions. The nonholonomic manipulator developed by Søerdalen et al. (1994) is an n-joint serial manipulator that can be controlled using only two independent actuators. What lies at the heart of both these robotic devices is a nonholonomic mechanical transmission device, involving rolling wheels in contact with a sphere, that effectively acts as a continuously variable transmission (CVT).

CVTs have been the object of considerable research interest within the mechanical design community, driven primarily by automotive applications. Unlike conventional stepped transmissions, a CVT allows for a continuous range of gear ratios that can, up to certain device-dependent physical limits, be selected independently of the applied torque. This feature of the CVT allows for engine operation at the optimum fuel consumption point, improving overall vehicle efficiency.

Existing CVTs can be classified into four types: belt drive, variable stroke drive, hydrostatic/dynamic drive, and
traction/friction drive (see report of U.S. Department of Energy (1982) for a description of the operating principles of each type). The last type of CVT transmits power through a contact between two rotating elements, with the drive ratio varied by controlling the effective onset radius of the contact point. Only traction/friction drives directly exploit the nonholonomy inherent in rolling contacts, and combine them with shifting mechanism designs that are based on angular momentum conservation (see Figure 1, by courtesy of DOE report 1982). Traction/friction drives also tend to be more compact, offering more flexibility in terms of their design and range of applications, and offer the possibility of precise positioning without backlash (Moronuki and Furukawa 1988; Kato 1990; Lee and Tomizuka 1996; Chang and Toumi 1998; Kurosawa, Takahashi, and Higuchi 1998).

While the kinematics of nonholonomic CVTs of the type used in the Cobot (Moore et al. 1999) and the nonholonomic manipulator (Søerdalen et al. 1994) are well-understood, by contrast very little attention has been given to the understanding of their dynamic and elastic behaviors, e.g., spin and other sources of power loss, as well as elasto-dynamic modeling. Clearly such an understanding is essential to making nonholonomic CVTs a practical, reliable, and widely used machine element for robotic systems.

1.2. Mobile Robot Design

In recent years there has been an explosion of research activity in mobile robots, driven in part by the proliferation of service robots designed for, e.g., patient transportation, security monitoring, mobile manipulator platforms, etc. In contrast to the extensive literature on mobile robot motion planning and control—traceable in large part to the mathematical richness of the nonholonomy problem—relatively little attention has been given to hardware platforms and other mechanical aspects of mobile robots (Jones and Flynn 1993; Borenstein, Everett, and Feng 1996).

While mobile robots vary widely in design, in general a wheeled mobile robot requires a minimum of two actuators for moving about in the plane, each with a dedicated controller; this is true both for wheel drives employing differential gears and differential drives. Mobile robots also typically rely on electric motors for actuation, in particular dc motors, because of their relatively simple control, and the fact that power can be supplied from battery sources. The latest dc motors possess motor drivers efficient enough to be used as variable speed drives (Leonhard 1996; Kassakian, Schlecht, and Verghese 1991). Despite these performance improvements, current mobile robots are still limited by battery lifetimes. It is also still true that dc motors are most efficient in the low torque, high speed operating regime (see Figure 2).

Reduction gears offer a means of operating a dc motor in its region of maximum efficiency, while simultaneously reducing the load torque and increasing motor speed according to the chosen gear ratio. Unlike automobiles, however, it is still impractical to equip mobile robots with conventional transmission devices given the manufacturing costs, size, and other practical limitations of current transmissions.

An increasingly practical alternative to reduction gears is the CVT. By allowing for an infinite range of gear ratios, a CVT allows the motor to deliver a range of torques while continuously operating at its most efficient speed. CVTs, provided they are sufficiently compact and have only minimal power loss, hold the potential of a mobile robot with improved efficiency, capable of complete planar accessibility with just a minimal number of actuators and mechanical components. Developing a mobile robot around a CVT, however, introduces a number of new design challenges. For example, there is the issue of how to design a simple pivoting device that eliminates the need for an actively controlled steering mechanism. Designing a stable controller that takes into account the dynamics of the complete system, and maximizes power efficiency, is also an issue.

1.3. Objectives of the Paper

In Kim et al. (2002), a novel type of nonholonomic CVT, the S-CVT, was proposed, together with a kinematic and dynamic analysis of the device. This paper reports on the development of a new type of wheeled mobile robot, referred to as MOSTS (Mobile Robot with a Spherical Transmission System), based on the S-CVT. Typical mobile robots achieve planar mobility by employing an additional controlled actuator, such as a steering wheel or a motor for differentiating each wheel velocity. MOSTS is a minimal design in the sense that, with the design of a novel pivoting device that takes advantage of the flexibility of the S-CVT, it can turn about its center and change its direction of movement without the need for a steering actuator and controller. We also present a nonlinear shifting controller based on feedback linearization that takes into account the dynamics of the S-CVT.

A second objective of this paper is to examine the sources of power loss in the S-CVT, in particular spin loss. Clearly spin loss strongly impacts the performance of all the nonholonomic CVT types mentioned above. However, an accurate analysis of the mechanism of slippage, particularly for spherical CVTs, is still an open research topic. In this paper we develop a friction model for the S-CVT, and perform a quantitative analysis of spin loss. Our approach generalizes in a straightforward way to more general spherical CVT designs such as the Cobot and nonholonomic manipulator, and serves as a theoretical tool in searching for ways to reduce spin loss.

To evaluate the energy efficiency of MOSTS and the performance of the S-CVT as a machine element, we perform experiments with a hardware prototype of MOSTS. The results are benchmarked numerically with a differential drive type mobile robot equipped with a reduction gear. Taken together, we hope our results contribute to narrowing the gap...
between idealized nonholonomic CVTs and their use as practical, reliable devices for robotic systems.

2. Nonholonomic Spherical CVT

In this section a new type of spherical continuously variable transmission (S-CVT) is described. The S-CVT is marked by its simple kinematic design and IVT characteristics, i.e., the ability to transition smoothly between the forward, neutral, and reverse states without the need for any brakes or clutches. Because the S-CVT transmits power via rolling resistance between metal on metal, it has limitations on the overall transmitted torque, which is effectively determined by the static coefficient of friction and the magnitude of the normal forces applied to the sphere. Due to this torque limitation, the S-CVT is not intended for automobiles and other large capacity power transmission applications. Target applications for the S-CVT include mobile robots, household electric appliances, small-scale machine tools, and other applications with moderate power transmission requirements. Although the current design of the S-CVT is based on friction drive designs, it is our expectation that the power capacity of the S-CVT can be increased by the use of traction oil, an issue which we do not pursue further in this paper.
Other spherical CVT structures have been proposed for use in passive mobile robots and for use as nonholonomic joints in robot manipulators. The Cobot (Moore et al. 1999) adopts a rotational CVT to provide smooth, hard virtual surfaces for passive haptic devices in place of conventional motors. Its rotational CVT consists of a sphere caged by four rollers, and adopts the joint speeds and task space speeds along with the steering angles as control inputs. Another application can be found in underactuated manipulators, designed by Steerdalen et al. (2001). This work proposes a new type of manipulator architecture using a CVT-type robot joint that takes advantage of the inherent nonholonomy of the CVT. Although these systems are designed to manipulate the speed ratio using a CVT mechanism, their main purpose is not for power transmission to improve the energy efficiency. Furthermore, the shifting mechanism of the S-CVT is quite different from these previous designs, as will be described below.

2.1. Structure

The S-CVT is composed of three pairs of input and output discs, variators, and a sphere (see Figure 3). The input discs are connected to the power source, e.g., an engine or an electric motor, while the output discs are connected to the output shafts. The sphere transmits power from the input discs to the output discs via rolling resistance between the discs and the sphere. The variators, which are connected to the shifting controller, are in contact with the sphere like the discs, and constrain the direction of rotation of the sphere to be tangent to the rotational axis of the variator. To transmit power from the discs to the sphere or from the sphere to the discs, a device that supplies a normal force to the surface, such as a spring or hydraulic actuator, must be installed on each shaft. As can be seen in Figure 3, the structure and components of the S-CVT are simple enough to allow for a considerable reduction in size and weight compared to conventional transmissions. The orientations of the input and output shafts can also be located freely using rollers at arbitrary positions rather than discs.

2.2. Kinematic and Dynamic Analysis of the S-CVT

When the input device is actuated by a power source, the input disc rotates about the input shaft. This rotation in turn causes a rotation of the sphere, due to the condition of rolling contact without slip between the input disc and the sphere. Rotation of the sphere in turn causes a rotation of the output discs, and subsequently of the output shaft. In the absence of any contact between the sphere and the variator, the axis of rotation of the sphere will largely be determined by an equilibrium condition among the various contact and load forces being applied to the sphere.

By varying the axis of rotation of the sphere, it is in turn possible to vary the radius of rotation of the contact point between the input disc and the sphere, \( R_c \), as well as the radius of rotation of the contact point between the output disc and the sphere, \( R_o \) (see Figure 4). In this way the speed-torque ratio of the S-CVT can be adjusted. Figure 4 shows the various alignments of the variator for the forward, neutral, and reverse states of the output shaft of the S-CVT. The neutral state, which corresponds to zero rotation of the output disc, is achieved when \( R_o \) becomes zero. As apparent from the figure, the forward, neutral, and reverse states can all be achieved by smoothly manipulating the variator alignment, without the need for any additional clutches or brakes.

Assuming roll contact without slip, the speed and torque ratio between the input and output discs is related to the variator angle by the following relations:

\[
\frac{\omega_{out}}{\omega_{in}} = r_o \tan \theta \tag{1}
\]

\[
\frac{T_{out}}{T_{in}} = r_o \cot \theta \tag{2}
\]

where \( \theta \) is the angular displacement of the variator, \( \omega_{in} \) and \( \omega_{out} \) are the respective angular velocities of the input and output shafts, \( T_{in} \) and \( T_{out} \) are the respective input and output torques, and \( r_i \) and \( r_o \) are the respective radii of the contact points of the input and output discs (see Figure 4).

Although ideally an infinite torque ratio is possible with the S-CVT, as seen in eq. (2), in practice there is a limit to the torque that can be transmitted because power transmission occurs from rolling resistance of metal on metal. The limiting torque \( T_{max} \) is determined by the static coefficient of friction \( \mu_s \) and the normal force \( N \) exerted by the output disc spring mechanism on the sphere according to the relation \( T_{max} = r \mu_s N \), where \( r \) is the contact radius of the disc. When either the input or output torque applied at the disc-sphere contact exceeds this limit, slippage can occur.

Using the free body diagrams of the sphere in Figure 5, the dynamic equations of the S-CVT can be derived as follows (Kim et al. 2002):

\[
\begin{bmatrix}
\frac{2(I_a + I_r + \epsilon m^2)}{2} & \frac{\epsilon m \omega_{in}^2}{2} \\
\frac{\epsilon m \omega_{in}^2}{2} & \frac{2 \epsilon m \omega_{in}^2}{2} \\
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\ddot{\phi} \\
\end{bmatrix}

= \begin{bmatrix}
F_s \\
F_c \cos \theta - F_s \sin \theta \\
\end{bmatrix}
\tag{3}
\]

where

- \( I_a \) = moment of inertia of the shifting actuator shaft and connecting rod [kg · m²],
- \( I_r \) = moment of inertia of the variator [kg · m²],
- \( I_s \) = moment of inertia of the sphere [kg · m²],
- \( m \) = mass of the variator [kg],
- \( \epsilon \) = eccentric distance between the centers of the shifting actuator shaft and variator [m],
- \( R \) = sphere radius [m],
- \( F_s \) = shifting force [N],
- \( F_c \) = control force [N].

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Fig. 3. Spherical CVT.

(a) Standard structure of S-CVT.  
(b) 3-dimensional view.

Fig. 4. Operating principles of S-CVT.

**UPPER VIEW**

**ISOMETRIC VIEW**

- **OPPOSITE ROTATIONAL DIRECTION**
- **NO OUTPUT ROTATION**
- **SAME ROTATIONAL DIRECTION**

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3. Spin Loss Analysis of the Spherical CVT

Spin loss and slip loss are the two main sources of power loss in traction and friction drives, excluding the losses due to the seals, thrust bearings, normal force loading device, and shifting actuator (particularly in traction drives, shifting actuator is composed of hydraulics). Slip loss is caused by rotational slippage at the contact points, resulting mainly from changes in the transmitted forces. When the exerted force from the driving element is larger than the static friction force, a rotational slippage occurs at the contact points, and the magnitude of the transmitted force is determined from the underlying friction mechanism. In this case the kinematic constraints of the S-CVT are no longer satisfied, and the dynamics of the overall system needs to be rederived taking into account the extra degrees of freedom that now become available. Predicting slip loss therefore requires information on velocities, accelerations, and forces at the contact points together with a much more complicated dynamic model of the system. Slip loss analysis is at best difficult and unreliable for the S-CVT, and we instead focus on spin loss in this paper.

The other source of loss, spin loss, is also one of the main design issues in traction drives. Spin loss results from the elastic contact deformation of rotating bodies that have different rotational velocities. Relative velocities resulting from elastic contact between rotating bodies usually give rise to friction mechanisms, in which case friction moments (spin loss) occur at the contact region. Spin loss is caused not by changes in the applied or exerted forces, but by the difference in rotating velocities and geometric properties of the bodies in contact. Spin loss (see Figure 6, Choi et al. 1999) is of particular concern in traction drives. To reduce spin loss in traction and friction drives, designers have investigated different approaches to optimal contact geometry design, applying normal loads, and controller design (Loewenthal et al. 1981; Wang and Fries 1989; Tanaka et al. 1989).

\[ F_i = \text{driving force delivered from the input discs [N]} \]
\[ F_o = \text{Reaction force caused by the output discs connected to the load torque [N]} \]
\[ \omega = \text{spinning rate of the sphere [rad/s]} \]
For nonholonomic spherical elements like S-CVT, spin loss is present whenever the rotational axis of the sphere is not parallel to the roller’s axis (Moore 1997). Although many efforts have been dedicated to compensate for slip in a variety of mechanical systems, an accurate analysis of the mechanism of slippage is still an open topic of research. Furthermore there is at present no theoretical framework for the analysis and quantification of spin loss for devices like the spherical CVT.

3.1. Modified Friction Model for the S-CVT

Contrary to the predictions derived from the classical stick-slip friction model, it has been experimentally found that small relative displacements between two bodies in contact occur when the applied tangential force is less than that produced by static friction (Courtney-Pratt and Eisner 1957). Dahl (1977) provided a model of this pre-sliding displacement phenomenon, known as the Dahl model, that assumes friction force is a function of displacement $x$ and time $t$; however, it cannot be applied when the velocity $\dot{x} \gg 0$. In addition, several researchers have found a source for this discrepancy in the Stribeck effect, and experimentally derived a model of friction variation with velocity.

In order to predict the spin loss of friction drives including the S-CVT, one must consider the pre-sliding effect in the vicinity of zero relative velocity. Friction models based on Dahl’s have difficulties in numerical integration due to the high stiffness and damping coefficients; moreover these coefficients must be obtained through a careful experimental analysis. For our purposes we develop a modified version of the classical stick-slip friction model that includes a Stribeck effect-like term:

$$F_r = \left[ (\mu_s - \mu_k) \exp \left( -\frac{\Delta V}{V_{m_r}} \right)^{\frac{1}{2}} + \mu_k \right] P \text{sgn}(\Delta V) \tag{4}$$

where $V_{m_r}$ is the critical Stribeck velocity, $\mu_s$, $\mu_k$ are static and kinetic coefficients of friction, $P$ the normal load, and $\Delta V$ the relative velocity, respectively. Here we neglect viscous friction as there is no lubricant layer in the S-CVT.

The friction force on the contact surface is determined by the normal force and friction coefficients. Considering that the kinetic coefficient of friction is related to the relative velocity $\Delta V$ between two rotating bodies, we first investigate the relative velocity field on the contact surface. In the following subsection, Hertzian results (Timoshenko and Goodier 1951; Chung and Han 1996) for elastic deflection are employed to construct the geometric parameters of the contact surface.

3.2. Disc-Sphere Contact

Figure 7(a) shows the contact surface between the sphere and upper variator. We set at the center of the contact surface $S$ a local coordinate frame $[\xi \eta]$ along the $xyz$ directions, with $\xi$ in the rolling direction. Supposing there is no bending deformation of the variator along the $x$ and $z$ axes, the normal deflection $\delta$ and contact surface radius $c$ can be calculated from Hertzian theory to be

$$c = 1.109 \times \sqrt{\frac{P}{E} R} \quad \delta = 2.64 \times \sqrt{\frac{P^2}{E^2}} \tag{5}$$

where $P$ is the normal force, $R$ the sphere radius, and $E$ represents the equivalent Young’s modulus.

To obtain each velocity field, we first recall that the sphere has a pure rotational speed of $\omega_z$ in the $z$ direction and the variator a rotational speed of $\omega_y$ in the $y$ direction. The velocity field of the sphere $V_i(\xi, \eta)$ and that of variator $V_j(\xi, \eta)$ on the contact surface can be obtained as follows:

$$V_i(\xi, \eta) = \left[ (R - \delta)\omega_z, 0 \right],
V_j(\xi, \eta) = \left[ (\epsilon + \eta)\omega_z, -\xi\omega_y \right], \tag{6}$$

where $\epsilon$ is the distance between the contact surface center and variator center. Consequently, the relative velocity field $\Delta V_i(\xi, \eta)$ can be derived as

$$\Delta V_i(\xi, \eta) = \left[ (R - \delta)\omega_z - (\epsilon + \eta)\omega_z, \xi\omega_y \right]. \tag{7}$$

The relative velocity fields at the other contact points (input and output discs) can be similarly obtained.

As shown in eq. (7), there is a relative velocity component of $\Delta V_i$ in the $\eta$ direction, $\Delta V_{i\eta}$, on the contact surface. However, $\Delta V_{i\eta}$ does not accumulate total relative velocity in the $\eta$ direction, because it is symmetric along the $\eta$ axis. Therefore, $\Delta V_{i\eta}$ contributes to spin along the direction normal to the $\xi\eta$ plane. In the case of $\Delta V_i$, the relative speed of $-\delta \omega_z + \eta \omega_y$ occurs in the rolling direction (recall the rotational speed relation of $R\omega = \epsilon \omega_z$). Note that $\delta$ becomes small enough to be neglected compared $R$ (for example, $\delta = 2.4 \times 10^{-3}$ mm, $c = 0.59$ mm, and $R = 30$ mm for the case of the S-CVT prototype); therefore $\Delta V_i$ can be approximated to be $-\eta \omega_y$. The contribution of $\Delta V_i$ is also a spin, similar to $\Delta V_{i\eta}$.

The vector diagram of relative velocity is obtained using typical values of $\omega_z, \omega_y, \epsilon, R, P$, and $E$ that correspond to the S-CVT prototype specification (see Figure 7(b)). The spin velocity field can be found straightforwardly, although there are no excessive forces that cause slippage. From this result, we can be assured that there must be spin in the contact surface around the origin of the local coordinate frame of the contact point in the S-CVT regardless of the existence of shear force resulting slippage. This relative velocity field in turn gives rise to spin moments. Moreover the resulted spin moment has a pure rotational speed of $\omega_y$ in the $y$ direction. Therefore, the relative velocity field $\Delta V_j(\xi, \eta)$ does not accumulate in the rolling direction and the amount of normal deflection $\delta$ is small enough to neglect compared to $R$.

Now we consider the normal pressure distribution on the contact patch to obtain the friction force in eq. (4). A Hertzian pressure distribution develops in the circular shaped contact
patch (with radius $c$) between the sphere and disc. The pressure $p$ at each point in the contact surface is known to be of the form

$$p(\xi, \eta) = \frac{3}{2} \frac{P}{\pi c^3} \sqrt{c^2 - \xi^2 - \eta^2}. \quad (8)$$

The maximal normal pressure $p_{\text{max}}$ is located at the center of the contact surface; at the boundaries, the normal pressure $p$ becomes zero. In the following subsection, we now perform a qualitative analysis of spin loss using the obtained results on relative velocity field and normal pressure distribution.

### 3.3. Quantitative Analysis of Spin Loss

Consider the infinitesimal area at the contact surface $S$, with the friction force of the $i$th area in the rolling direction ($\xi$) denoted $F_{\xi i}$, and $F_{\eta i}$ the force in the $\eta$ direction. The total friction forces $F_{\xi}$ and $F_{\eta}$ can be obtained using the following equations:

$$ F_{\xi} = \int_{-c}^{c} \int_{-c}^{c} F_{\xi i}(\xi, \eta) \, d\xi \, d\eta, \quad F_{\eta} = \int_{-c}^{c} \int_{-c}^{c} F_{\eta i}(\xi, \eta) \, d\xi \, d\eta. $$

Recall that the normal pressure distribution has symmetries along the $\xi$ and $\eta$ axes, and that $\Delta V_\xi$ in eq. (7) varies along the $\eta$ direction (neglecting $\delta$), and $\Delta V_\eta$ along the $\xi$ direction; there are no total relative velocities in the $\xi$ and $\eta$ directions. Therefore one can conclude that $F_{\xi}$ and $F_{\eta}$ become zero, and that these forces do not cause any rolling directional slippage resulting in slip loss.

Using the proposed friction model in eq. (4), the spin moment $T_{\text{spin}}$ for the variator can be calculated as

$$ T_{\text{spin}} = \frac{3}{4\pi} \int_{-c}^{c} \int_{-c}^{c} (\eta F_{\xi i} + \xi F_{\eta i}) \, d\eta \, d\xi \quad (9) $$

where

$$ F_{\xi i} = \left[ (\mu_s - \mu_k) \exp \left(-\frac{\left(\frac{\Delta V_\xi}{V_{\text{str}}}ight)^2}{2} \right) + \mu_k \right] p(\xi, \eta) \text{sgn}(\Delta V_\xi), $$

$$ F_{\eta i} = \left[ (\mu_s - \mu_k) \exp \left(-\frac{\left(\frac{\Delta V_\eta}{V_{\text{str}}}ight)^2}{2} \right) + \mu_k \right] p(\xi, \eta) \text{sgn}(\Delta V_\eta), $$

$$ \Delta V_\xi = -\eta \omega_v, \quad \Delta V_\eta = \xi \omega_v. $$

Rearranging and integrating by parts, eq. (9) becomes

$$ T_{\text{spin}} = \frac{3Pc}{4\pi} \left( \mu_i + (\mu_s - \mu_k) \right) \frac{2V_{\text{str}}^2}{\omega_v^2} \left[ 1 - \exp \left( -\left(\frac{c \omega_v}{V_{\text{str}}}ight)^2 \right) \right] \left[ \frac{c^2}{\omega_v^2} + \frac{V_{\text{str}}^2}{\omega_v^2} \right] $$

$$ \left\{ \begin{array}{l} \mu_i + (\mu_s - \mu_k) \frac{2V_{\text{str}}^2}{\omega_v^2} \left[ 1 - \exp \left( -\left(\frac{c \omega_v}{V_{\text{str}}}ight)^2 \right) \right] \left[ \frac{c^2}{\omega_v^2} + \frac{V_{\text{str}}^2}{\omega_v^2} \right] \end{array} \right\} $$

$$ \left\{ \begin{array}{l} \mu_i + (\mu_s - \mu_k) \frac{2V_{\text{str}}^2}{\omega_v^2} \left[ 1 - \exp \left( -\left(\frac{c \omega_v}{V_{\text{str}}}ight)^2 \right) \right] \left[ \frac{c^2}{\omega_v^2} + \frac{V_{\text{str}}^2}{\omega_v^2} \right] \end{array} \right\} $$

where $c$ is the radius of the contact surface, which can be calculated using eq. (5).

To investigate the amount of spin loss at the contact points of the S-CVT, we calculate the respective spin losses using eq. (10) for the input and output discs and variators with typical values for $\mu_i$, $\mu_r$, $V_{\text{str}}$, $P$. Figure 8 shows the numerical results for spin loss and the rotational speeds of the input/output discs and variator at a constant input speed of 3000 rpm with

Fig. 7. Relative velocity on S-CVT.
increased relative velocity reduces the friction force. Is helpful to operate the S-CVT with high input speeds; the ratio of spin loss becomes much greater. To reduce this loss, one can note that the ratio of spin loss to static friction torque is almost 3.67%. Considering the input torque corresponding to static friction to kinetic friction.

Relative velocity in turn reduces the friction force from that occurring the relative velocity $\nu_{\Delta} \partial V$. Increasing the input speed increases the rotational speed of variator, causing the relative velocity $\Delta V$ to become large. The increased relative velocity in turn determines the input/output power ratio. When the relative velocity in turn reduces the friction force from that occurring the relative velocity $\nu_{\Delta} \partial V$. Increasing the input speed increases the rotational speed of variator, causing the relative velocity $\Delta V$ to become large. The increased relative velocity in turn determines the input/output power ratio.

The central idea is to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied. We first present a stability analysis of the shifting system, followed by the tracker design and a numerical analysis of its performance.

4. Shifting Controller for the S-CVT

The shifting controller realizes the target gear ratio, which in turn determines the input/output power ratio. When the shifting command for a certain gear ratio is given, the shifting system must be stabilized so as to realize the demanded gear ratio with the desired performance (e.g., minimum shifting effort, short settling time, etc.). The shifting command for a CVT can be either a final value or a trajectory of the target gear ratio. The shifting controller is denoted a stabilizer (or regulator) in the former case and tracker (or servo) in the latter case.

Because the shifting system of the S-CVT has second-order nonlinear dynamics, and the original open-loop system reveals unstable characteristics, in this section, we develop a feedback controller based on exact feedback linearization. Lyapunov’s (indirect) linearization method is involved with the local stability of a nonlinear system. It is a formalization of the intuition that a nonlinear system should behave similarly to its linearized approximation for small range motions. Because all physical systems are inherently nonlinear, Lyapunov’s linearization method serves as the fundamental justification of using linear control techniques in practice, i.e., that stable design by linear control guarantees the stability of the original physical system locally.

We first investigate the local stability of the S-CVT shifting system using Lyapunov’s linearization method. We first recast the shifting dynamics of eq. (3) into state-space form. Letting $x_1 = \theta$, $x_2 = \dot{\theta}$ be the states, the corresponding state-space equation assumes the following form:

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{D} [a_{22} F_i - a_{12} (F_i \cos x_1 - F_o \sin x_1)] ,
\end{align*}$$

where

$$\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} = \begin{bmatrix}
2 \frac{\Delta x + x m \epsilon}{r} & 2 \frac{\epsilon}{r} \\
\frac{\Delta x}{r} & 2 \frac{\epsilon}{r} + 2 \frac{\Delta x m \epsilon}{r}
\end{bmatrix} ,
$$

$$D = a_{11} a_{22} - a_{12} a_{21} .$$

Using the trigonometric transformation

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \phi) , \quad \phi = \tan^{-1} \left( \frac{b}{a} \right) ,$$

eq (11) can be written

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{D} \{a_{22} \sqrt{F_i^2 + F_o^2 \sin(x_1 - \phi) + a_{22} F_i} \}
\end{align*}$$

where

$$\phi = \tan^{-1} \left( \frac{F_i}{F_o} \right) .$$

Noting that $D = \frac{2 I_s (I_o + I_s m^2)}{\epsilon R} + \frac{4 R I_s (I_o + m \epsilon^2)}{\epsilon^3} \frac{\epsilon^3}{R}$ is always larger than zero, the equilibrium point is given by

$$x_1^* = \phi = \tan^{-1} \left( \frac{F_i}{F_o} \right) , \quad x_2^* = 0 , \quad F_i^* = 0 .$$

Fig. 8. Spin losses on S-CVT at input speed of 3000 rpm.
The equilibrium point of interest is \( x^* = (\phi, 0) \). Physically, this point corresponds to steady state of the shifting system in which no shifting occurs.

The Jacobian matrix \( J \) of the shifting system (12) linearized about the equilibrium point becomes

\[
J = \begin{bmatrix}
\frac{a_{12}}{D} \sqrt{F_i^2 + F_o^2} & 1 \\
0 & 0
\end{bmatrix}.
\]

(14)

The eigenvalues of \( J \) are \( \lambda_i = \pm \frac{a_{12}}{D} \sqrt{F_i^2 + F_o^2} \), indicating that the linearized system is unstable at this equilibrium point. Physically this implies that when the input-output force relation \( (F_i \cos \theta = F_o \sin \theta) \) is broken (i.e., steady state is destroyed) by some disturbances from the input or output force, maintaining stability requires a change in variator angle (gear ratio) from the shifting actuator, or by a change in input force from the power source controller. In order to make the shifting system stable, one can conclude that an appropriate feedback controller is necessary. We now discuss the design of a feedback controller based on the exact feedback linearization method.

### 4.2. Input-State Linearization

Consider an affine single-input nonlinear system

\[
\dot{x} = f(x) + g(x) \cdot u,
\]

(15)

where \( x \in \mathbb{R}^n, u \in \mathbb{R} \). The objective of input-state linearization is to find a transformation \( z = T(x) \) and a control of the form \( u = x + \beta(x) \cdot \nu \) such that

\[
\begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= z_3, \\
& \vdots \\
\dot{z}_n &= \nu.
\end{align*}
\]

(16)

We refer the reader to, e.g., Khalil (1996) and Slotine and Li (1991) for a more in-depth treatment of feedback linearization, including the conditions for the existence of a solution. For our purposes the vector fields \( f \) and \( g \) of the shifting system are given by

\[
\begin{align*}
f &= \begin{bmatrix}
x_2 \frac{a_{13}}{D} \sqrt{F_i^2 + F_o^2} \sin(x_1 - \phi) \\
0 \frac{a_{22}}{D}
\end{bmatrix}^T, \\
g &= \begin{bmatrix}
0 \frac{a_{22}}{D}
\end{bmatrix}^T.
\end{align*}
\]

(17)

The Lie bracket \( ad_f g \) of two vector fields \( f \) and \( g \) is defined to be

\[
ad_f g = \nabla g \cdot f - \nabla f \cdot g,
\]

which is also often denoted \( [f, g] \). Repeated Lie brackets can be defined recursively according to

\[
\begin{align*}
ad_{g^k} g &= g, \\
ad_{g^{k+1}} g &= [f, ad_{g^k} g],
\end{align*}
\]

With these preliminaries, input-state linearization of the shifting system can be performed via the following steps:

1. Construct the vector fields \( g, ad_f g, \ldots, ad_f^{n-1} g \) for our system.
2. Check the controllability and involutivity conditions.
3. Find the first new state \( z_1 \) from

\[
\nabla z_1 \cdot ad_f g = \begin{bmatrix} 0 \end{bmatrix}, \\
\nabla z_1 \cdot ad_f^{k-1} g \neq \begin{bmatrix} 0 \end{bmatrix}, k = 0, 1, \ldots, n - 2, \\
\nabla z_1 \cdot ad_f^{k-1} g = \begin{bmatrix} 0 \end{bmatrix}.
\]

(18)

4. Compute the diffeomorphism that transforms the state \( x \) into the new state \( z, T(x) = [z_1, L_1 z_1 \ldots L_1^{n-2} z_1]^T \), and the input transformation using

\[
\begin{align*}
\alpha(x) &= -\frac{L_1^2 z_1}{L_1 L_1^{n-2} z_1}, \\
\beta(x) &= \frac{1}{L_1 L_1^{n-2} z_1}.
\end{align*}
\]

(19)

Knowing that the system order \( n = 2 \) and \( \nabla g = 0, \) the corresponding Lie bracket becomes

\[
\begin{align*}
ad_{ad_f g} g &= \nabla g \cdot f - \nabla f \cdot g \\
&= 0 - \begin{bmatrix} a_{12} \sqrt{F_i^2 + F_o^2} \cos(x_1 - \phi) & 1 \end{bmatrix} \begin{bmatrix} 0 \\
\frac{a_{22}}{D}
\end{bmatrix} \\
&= \begin{bmatrix} -\frac{a_{22}}{D} & 0 \end{bmatrix}^T.
\end{align*}
\]

(20)

A simple check of the rank of the controllability matrix

\[
\begin{bmatrix} g & ad_f g & ad_f^{2} g & \ldots & ad_f^{n-1} g \end{bmatrix} = \begin{bmatrix} 0 & -\frac{a_{22}}{D} \\
\frac{a_{22}}{D} & 0
\end{bmatrix},
\]

(21)

which is full rank, confirms that the shifting system is input-state linearizable.

We now find the diffeomorphism \( T(x) \) that transforms the original shifting dynamics into a linear system. Using the results of eq. (18), the necessary conditions for the first state \( z_1 \) are

\[
\begin{align*}
\frac{\partial z_1}{\partial x_1} &\neq 0, \\
\frac{\partial z_1}{\partial x_2} &= 0.
\end{align*}
\]

Thus \( z_1 \) must be a function of \( x_1 \) only. Among the various candidates for \( z_1, \) the simplest solution is \( z_1 = x_1 - \phi. \) The other state can be obtained from \( z_1 \) as

\[
z_2 = \nabla z_1 f = x_2.
\]
The corresponding diffeomorphism $T(x)$ can be obtained as
\[ z = T(x) = \begin{bmatrix} x_1 - \phi \\ x_2 \end{bmatrix}. \tag{22} \]

Accordingly the input transformation in eq. (19) is
\[ u = \frac{v - \nabla z_{2} f}{\nabla z_{2} g}, \]
which can be written explicitly as
\[ u = \frac{D}{a_{22}} \{ v - \frac{a_{12}}{D} \sqrt{F_{1}^{2} + F_{2}^{2} \sin(x_{1} - \phi)} \}. \tag{23} \]

As a result of the above state and input transformations, we end up with the following set of linear equations:
\[ \dot{z}_{1} = z_{2}, \quad \dot{z}_{2} = v \tag{24} \]
\[ v = \frac{a_{12}}{D} \sqrt{F_{1}^{2} + F_{2}^{2} \sin(x_{1} - \phi)} + \frac{a_{22}}{D} u, \tag{25} \]
thus completing the input-state linearization.

### 4.3. Shifting Controller Design

Using the above input-state linearization results, we now design a shifting controller for tracking. In this case, it is desired to have the variator angle $\theta$ track a prescribed trajectory $\dot{\theta}_d$. The input $v$ is chosen to be of the form
\[ v = \dot{z}_{id} - k_{1} \epsilon - k_{2} \dot{\epsilon}, \tag{26} \]
where $\epsilon = z_{1} - z_{id}$ and $z_{id} = \dot{\theta}_d - \phi$. The associated error dynamics is of the form with the gain values $k_{1}$ and $k_{2}$ chosen appropriately to ensure stability.
\[ \ddot{\epsilon} + k_{1} \ddot{\epsilon} + k_{2} \epsilon = 0. \tag{27} \]

The resulting closed-loop dynamics of the shifting system (27) can be viewed as the canonical form of a general second-order oscillation problem:
\[ s^2 + 2 \zeta \omega_n s + \omega_n^2 = 0. \]

Hence one can give physical meaning to the feedback gains as the respective damping ratio $\zeta$ and the natural frequency $\omega_n$. The relation between the feedback gains and $\zeta, \omega_n$ are simply
\[ k_{1} = \omega_n^2, \quad k_{2} = 2 \zeta \omega_n. \tag{28} \]

In this study, we desire our shifting controller to provide the most rapid response according to the shifting command without overshoot; we designate the settling time of the shifting system (the time in reaching the new equilibrium state) to be less than 1 s. Hence, we select the system damping ratio $\zeta$ to 1, which corresponds to the case of critical damping. For a given initial excitation, a critically damped system tends to approach the equilibrium position the fastest without any overshoot. Moreover, these feedback gains guarantee the asymptotic stability and tracking performance of the S-CVT shifting system. We consider two cases of $k_{1}, k_{2}$ (see Table 1).

Simulation results for the tracking controller are given for two sets of gain values. For the reference trajectory of the variator angle, we consider a sinusoidal function $\frac{\pi}{3} \sin(\frac{\pi}{2} t - \phi)$ (see Figure 9(a)), with the initial states of the system chosen as
\[ \theta = 42^\circ, \dot{\theta} = 0, \quad F_{i} = 1, \quad F_{o} = 1. \]

Maintaining the input-output force as the initial values, the calculated variator angle changes are depicted in Figure 9(b). As expected, both cases of feedback gains show asymptotic convergence of tracking error. The relevant tracking error and corresponding shifting effort are shown in Figure 10. Note that the time to reach the zero tracking error for case A is almost 0.7 s while for case B is almost 1.2 s. Thus we select the feedback gains for case A.

### 5. An S-CVT based Mobile Robot: MOSTS

In this section we present the design and prototype construction of an S-CVT based mobile robot, MOSTS (Mobile Robot with a Spherical Transmission System), including numerical and experimental results on its performance.

#### 5.1. Pivot Device for Planar Accessibility

In typical mobile robot designs, an additional controlled actuator, such as a steering wheel or a motor for differentiating each wheel velocity, is necessary in order to move a mobile robot in the plane. Employing a novel pivot device, however, we can eliminate the need for an additional steering actuator and controller. To change its heading direction, MOSTS turns about its center (or pivots) by rotating one of the wheels in the reverse direction. For this to occur, we have been inspired by the fact that the S-CVT can locate arbitrarily the orientation of an output shaft. To achieve this, it is necessary to locate one of the output shafts on the opposite side of the sphere (see Figure 11(a)).

To achieve this operation we have adopted an internal gear driven by a simple actuator (see Figure 11(b)), e.g., a limit switch used in automated windows, and an uncontrolled

<table>
<thead>
<tr>
<th>Table 1. Candidates for $k_{1}, k_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
</tr>
<tr>
<td>$k_{1} = 100, k_{2} = 20$</td>
</tr>
<tr>
<td>Case B</td>
</tr>
<tr>
<td>$k_{1} = 50, k_{2} = 10\sqrt{2}$</td>
</tr>
</tbody>
</table>
Fig. 9. Tracking performance of the S-CVT shifting system.

(a) Reference shifting command $\theta_d$.

(b) Variator angle changes.

Fig. 10. Tracking error and corresponding control.

(a) Tracking error.

(b) Control.
motor. Using simple analog devices, we build a pivot switch that can be turned off according to a pre-set current limit.

During pivot motion, each wheel rotates in opposite directions with the same magnitude, while the driving motor rotates continuously without any changes of state. The amount of pivot angle is determined by the amount of angular displacement of each wheel, which is controlled by the shifting actuator, or variator. Moreover, if a controlled actuator is used to rotate the movable output shaft, steering motion can be obtained. Designed in this fashion, MOSTS has the capability to move in the plane with one drive motor, one controller for the S-CVT, and one switching actuator.

5.2. Prototype Design

For the construction of the mobile robot platform, we have set the following performance targets:

1. A top speed of 5 m/s;
2. A maximum ascending angle of 10°;
3. A combined vehicle-payload mass of 50 kg.

From the above hardware specifications and material properties of the S-CVT, we choose a specific dc motor that produces a power of 150 W with 12 V under nominal operating conditions as the driving motor (see the details provided in Table 2).

The body of the mobile robot is designed to have a cylindrical shape, and a caster wheel is added to provide stable support. The internal body consists of three layers: a mechanical base for the transmission system, an intermediate layer for the battery pack and controller, and a top layer for peripherals and accessories, e.g., navigation sensors, manipulators. Rotary encoders sensing the speeds of the input and output shafts are also included. The overall size of the platform is 260 mm in radius, and 500 mm in height (see Figure 12).

5.3. Numerical and Experimental Results

In this subsection, we present numerical and experimental results that demonstrate the operation of MOSTS, and the energy savings possible from the use of the S-CVT mechanism over standard reduction gears. The reference path is shown in Figure 13(a); there are three linear movements and two pivot motions during 22 s. The distance traversed by the robot is 20 m. During the pivot motion, there is an auxiliary 2 s period for actuating the pivot switch, which is necessary to move one of output shafts of the S-CVT to the opposite direction. With this reference trajectory, we calculate the necessary wheel velocity profile satisfying the time constraints by using a sine

Table 2. DC Motor Characteristic Coefficients of MOSTS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rated voltage</td>
<td>12 V</td>
</tr>
<tr>
<td>Rated power</td>
<td>150 W</td>
</tr>
<tr>
<td>Motor-torque constant</td>
<td>0.0164 Nm/A</td>
</tr>
<tr>
<td>Back emf constant</td>
<td>0.0164 Volt/s/rad</td>
</tr>
<tr>
<td>Rotor winding resistance</td>
<td>0.117Ω</td>
</tr>
<tr>
<td>Stall torque</td>
<td>2.03 Nm</td>
</tr>
</tbody>
</table>
Fig. 12. Photograph of MOSTS and S-CVT prototype.

(a) Designed moving path.

(b) Calculated wheel velocity profile.

Fig. 13. Motion simulation of MOSTS.
function (see Figure 13(b)). The pivot motions in the path are specified as a 90° counter-clockwise rotation, followed by a 90° clockwise rotation.

First, we calculate the value of the output speed acceleration from the driving pattern under the assumption that the input voltage is held constant at 12 V. The exerted load torque is set to 2.5215 Nm, and the equivalent inertia with respect to the motor shaft is set to 0.01 km². With these values and the output speed, we extract the necessary variator angle \( \theta \) using a computed torque control algorithm. Finally, the trajectory of the variator angle is presented in Figure 14.

5.3.1. Numerical Results

Using the equations derived in the previous section, we have developed a simulation program that computes the motor speed, produced torque, and the power consumption. We use the Runge-Kutta fourth-order algorithm for numerical integration in the simulation program.

In Figure 15(a), the initial motor speed is about 7000 rpm, which is obtained from the no-load condition of the dc motor considered here. During the whole operation period, the motor speed varies freely between 6500 rpm and 7000 rpm through the entire stop, start, and pivot motions; this is almost the nominal speed with a no-load condition. With these values and the output speed, we extract the necessary variator angle \( \theta \) using a computed torque control algorithm. Finally, the trajectory of the variator angle is presented in Figure 14.

The calculated total energy consumption is 1389.61 J for the differential drive with reduction gear unit and 727.86 J for our CVT-based mobile robot. Simulation results indicate that our mobile robot with an S-CVT consumes less than 47.6% of the energy consumed by the differential drive, a significant improvement in energy efficiency.

5.3.2. Experimental Results

Using the sequential manipulation of the variator angle according to calculated values of Figure 14, we experimentally determined the actual energy consumption of MOSTS under the same reference trajectory mentioned above. The actual energy consumption is 1294.92 J, which is larger than the ideal case by 567.06 J. However, this is still smaller than the calculated energy consumption of 1389.61 J for the differential drive case (the actual energy consumption for this case will most likely be significantly higher than the calculated ideal rate). To investigate the reason behind this difference in total energy consumption, the induced motor current and the actual motor speed for the first two seconds are depicted in Figure 16. As the reference motion trajectory considered here has five repetitive sequences, it is sufficient to investigate the first two second period experimental results.

Observe that the initial motor current is almost 4 A, whereas the ideal value is almost zero. This initial induced motor current is mainly due to the power loss resulting from manufacturing errors including bearing friction, gear backlash, etc. Consequently, this power loss makes the driving motor run at lower speeds, causing the overall power efficiency to decrease. Actually, adopting the S-CVT into a mobile robot may increase the robot’s weight adversely. However, our mobile robot still holds the possibility of energy savings taking into account that the repeated start and stop motions of a mobile robot, in which large torques (requiring the large motor current) are applied from the driving motor, happen frequently.
Fig. 15. Motor behaviors of MOSTS.

Table 3. Energy Consumption; MOSTS vs. Differential Drive

<table>
<thead>
<tr>
<th></th>
<th>MOSTS</th>
<th>Differential drive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Simulation</td>
<td>Simulation</td>
</tr>
<tr>
<td></td>
<td>result</td>
<td>result</td>
</tr>
<tr>
<td></td>
<td>727.86 J</td>
<td>1389.61 J</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>Experimental</td>
</tr>
<tr>
<td></td>
<td>1294.92 J</td>
<td>??</td>
</tr>
</tbody>
</table>

Fig. 16. Experimental results.
6. Conclusion

In this paper, we have presented the design of a CVT-based mobile robot using a minimal number of actuator and control components, taking advantage of the performance features of the S-CVT. Such a CVT-based mobile robot has the advantage of being able to operate the motors in their regions of maximum efficiency, thereby prolonging the total run time of the robot. The addition of a novel pivot device also enables the mobile robot to achieve steering (more precisely, changing its heading direction) by using only a single drive motor and controller, unlike most existing mobile robot platforms. We examine spin loss in the S-CVT by first developing a modified friction model based on Hertzian elasticity theory that includes Striebeck effects, and perform a quantitative analysis that is generalizable to other nonholonomic CVT devices. A shifting controller is developed for the S-CVT based on input-state linearization of the dynamic equations. We also perform an in-depth analysis of the energy efficiency of our mobile robot taking into account features of the dc motors, the S-CVT, and the mobile robot dynamics. The results are benchmarked numerically with a differential drive type mobile robot equipped with a reduction gear. Furthermore, we perform an experiment using the prototype robot to verify the robot’s operation and the CVT characteristics. The numerical and experimental results show that our mobile robot with an S-CVT consumes less power than differential drive type robots, and suggests the S-CVT can be a useful nonholonomic machine element for a wide variety of robotics systems.

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References


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