Chapter 9

Dynamics of Open Chains

According to Newton's second law of motion, any change in the velocity of a rigid body is caused by external forces and torques. In this chapter we study once again the motion of open chain robots, but this time taking into account the forces and torques that cause it; this is the subject of **robot dynamics**. The associated dynamic equations—also referred to as the **equations of motion**—are a set of second-order differential equations of the form

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}), \tag{9.1}$$

where $\theta \in \mathbb{R}^n$ is the vector of joint variables, $\tau \in \mathbb{R}^n$ is the vector of joint forces and torques, $M(\theta) \in \mathbb{R}^{n \times n}$ is a symmetric and invertible matrix called the **mass matrix**, and $b(\theta, \dot{\theta}) \in \mathbb{R}^n$ are bias forces that lump together centrifugal and Coriolis, gravity, friction and other force terms that depend on θ and $\dot{\theta}$. One should not be deceived by the apparent simplicity of these equations; even for "simple" open chains all of whose joint axes are either orthogonal or parallel to each other, $M(\theta)$ and $b(\theta, \dot{\theta})$ can be extraordinarily complex.

Just as a distinction was made between a robot's forward and inverse kinematics, it is also customary to distinguish between a robot's **forward** and **inverse dynamics**. From the perspective of generating and simulating entire motion trajectories, it is useful to regard the robot dynamics as an input-output system, in which the inputs are torque trajectories $\tau(t)$, and the outputs are joint trajectories $\theta(t)$. From this perspective, in the case of forward dynamics the objective is to determine, from a given input joint torque trajectory $\tau(t)$ and appropriate set of boundary conditions on θ and $\dot{\theta}$, the output joint trajectory $\theta(t)$; this is usually done by numerically integrating Equation (9.1). In the case of inverse dynamics, the objective is to determine the joint torque trajectory $\tau(t)$ that generates some desired joint motion trajectory $\theta(t)$.

Slight variations in these interpretations of the forward and inverse dynamics are possible. For the inverse dynamics, observe that the velocity $\dot{\theta}$ and acceleration $\ddot{\theta}$ can be obtained by taking derivatives of the desired joint trajectory $\theta(t)$. Thus, given values for $(\theta, \dot{\theta}, \ddot{\theta})$ at time t, the joint torques τ can be obtained just by algebraic evaluation of the right-hand side of (9.1). This evaluation is also commonly referred to as the inverse dynamics. In the case of forward dynamics, since $M(\theta)$ is always invertible, Equation (9.1) can be rewritten

$$\ddot{\theta} = M^{-1}(\theta)\left(\tau - b(\theta, \dot{\theta})\right). \tag{9.2}$$

This evaluation of $\ddot{\theta}$ from given values for τ, θ, and $\dot{\theta}$ is also often referred to as the forward dynamics. While this interpretation may seem somewhat different from the previous one, in fact it is not: given an input torque trajectory $\tau(t)$ together with initial values for θ and $\dot{\theta}$ at $t = t_0$, forward integration of Equation (9.2) from $t = t_0$ then produces the complete output trajectory $\theta(t)$.

A robot's dynamic equations are typically derived in one of two ways: by a direct application of Newton's and Euler's dynamic equations for a rigid body (often called the **Newton-Euler formulation**), or from the **Lagrangian dynamics** formulation. The Lagrangian formalism is conceptually elegant and quite effective for robots with simple structures, e.g., with three or fewer degrees of freedom. However, the calculations can quickly become intractable for robots with more degrees of freedom. For general open chains, the Newton-Euler formulation leads to efficient recursive algorithms for both the inverse and forward dynamics that can also be assembled into closed-form analytic expressions for, e.g., the mass matrix $M(\theta)$ and other terms in the dynamics equation (9.1).

In this chapter we study both the Lagrangian and Newton-Euler dynamics formulations for an open chain robot. We conclude with a formulation of the dynamics in task space coordinates, or **operational space dynamics**.

9.1 Lagrangian Formulation

9.1.1 Basic Concepts and Motivating Example

The first step in the Lagrangian formulation of dynamics is to choose a set of independent coordinates $q \in \mathbb{R}^n$ that describes the system's configuration, similar to what was done in the analysis of a robot's configuration space. The coordinates q are called **generalized coordinates**. Once generalized coordinates have been chosen, these then define another set of coordinates $f \in \mathbb{R}^n$ that are dual to q, called **generalized forces**. f and q are dual to each other in the sense that their inner product $f^T q$ corresponds to work. A Lagrangian function $\mathcal{L}(q, \dot{q})$ is then defined as the overall system's kinetic energy minus the potential energy. The equations of motion can now be expressed in terms of the Lagrangian as follows:

$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}, \tag{9.3}$$

These equations are also referred to as the **Euler-Lagrange equations with external forces**[1]

We illustrate the Lagrangian dynamics formulation through two examples. In the first example, consider a particle of mass $\mathfrak{m}$ constrained to move on

[1]The external force f is zero in the standard form of the Euler-Lagrange equations.

a vertical line. The particle's configuration space is this vertical line, and a natural choice for generalized coordinate is the height of the particle, which we denote by the scalar variable $x \in \mathbb{R}$. Suppose the gravitational force $\mathfrak{m}\mathfrak{g}$ acts downward, and an external force f is also applied upward. By Newton's second law, the equation of motion for the particle is

$$f - \mathfrak{m}\mathfrak{g} = \mathfrak{m}\ddot{x}. \tag{9.4}$$

We now apply the Lagrangian formalism to this particle. The kinetic energy is $\mathfrak{m}\dot{x}^2/2$, the potential energy is $\mathfrak{m}\mathfrak{g}x$, and the Lagrangian is

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}\mathfrak{m}\dot{x}^2 - \mathfrak{m}\mathfrak{g}x. \tag{9.5}$$

The equations of motion are then given by

$$f = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \mathfrak{m}\ddot{x} + \mathfrak{m}\mathfrak{g}, \tag{9.6}$$

which matches Equation (9.4).

We now derive the dynamic equations for a planar 2R open chain moving in the presence of gravity. The chain moves in the x-y plane, with gravity acting in the $-y$ direction. Before the dynamics can be derived, the mass and inertial properties of all the links must be specified. To keep things simple the two links are modeled as point masses $\mathfrak{m}_1$ and $\mathfrak{m}_2$ concentrated at the ends of each link. The position and velocity of the mass of link 1 are then given by

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix} \\ \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} &= \begin{bmatrix} -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \end{bmatrix} \dot{\theta}_1, \end{aligned}$$

while that of the link 2 mass are given by

$$\begin{aligned} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} L_1 \cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ L_1 \sin\theta_1 + L_2\sin(\theta_1+\theta_2) \end{bmatrix} \\ \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} -L_1 \sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1 \cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}. \end{aligned}$$

Choose the joint coordinates $\theta = (\theta_1, \theta_2)$ to be the generalized coordinates. The generalized forces $\tau = (\tau_1, \tau_2)$ then correspond to joint torques (since $\tau^T\dot{\theta}$ must correspond to work). The Lagrangian $\mathcal{L}(\theta, \dot{\theta})$ is of the form

$$\mathcal{L}(\theta, \dot{\theta}) = \sum_{i=1}^{2} \mathcal{K}_i - \mathcal{P}_i, \tag{9.7}$$

where the link kinetic energy terms $\mathcal{K}_1$ and $\mathcal{K}_2$ are

$$\begin{aligned}
\mathcal{K}_1 &= \frac{1}{2}\mathfrak{m}_1(\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2}\mathfrak{m}_1 L_1^2 \dot{\theta}_1^2 \\
\mathcal{K}_2 &= \frac{1}{2}\mathfrak{m}_2(\dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}\left(\mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\dot{\theta}_1^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2 + L_2^2\dot{\theta}_2^2\right),
\end{aligned}$$

and the link potential energy terms $\mathcal{P}_1$ and $\mathcal{P}_2$ are

$$\begin{aligned}
\mathcal{P}_1 &= \mathfrak{m}_1\mathfrak{g}L_1\sin\theta_1 \\
\mathcal{P}_2 &= \mathfrak{m}_2\mathfrak{g}(L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2)).
\end{aligned}$$

The Euler-Lagrange equations (9.3) for this example are of the form

$$\tau_i = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_i} - \frac{\partial\mathcal{L}}{\partial\theta_i}, \quad i = 1, 2. \tag{9.8}$$

The dynamic equations for the 2R planar chain follow from explicit evaluation of the right-hand side of (9.8) (we omit the detailed calculations, which are straightforward but tedious):

$$\begin{aligned}
\tau_1 &= \left((\mathfrak{m}_1 + \mathfrak{m}_2)L_1^2 + \mathfrak{m}_2(2L_1L_2\cos\theta_2 + L_2^2)\right)\ddot{\theta}_1 \\
&\quad + \mathfrak{m}_2(L_1L_2\cos\theta_1 + L_2^2)\ddot{\theta}_2 - 2\mathfrak{m}_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2 - \mathfrak{m}_2L_1L_2\dot{\theta}_2^2\sin\theta_2 \\
&\quad + (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2) \\
\tau_2 &= \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_1 + \mathfrak{m}_2L_2^2\ddot{\theta}_2 - \mathfrak{m}_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2 \\
&\quad - \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2).
\end{aligned}$$

In the Lagrangian formulation of dynamics, once a set of generalized coordinates has been chosen, it is conceptually straightforward to formulate the Lagrangian, and from there to arrive at the dynamic equations by taking partial derivatives of the Lagrangian. In practice, however, the calculations can very quickly become intractable, especially as the degrees of freedom increase.

9.1.2 General Formulation

We now describe the Lagrangian dynamics formulation for general n-link open chains. The first step is to select a set of generalized coordinates $\theta \in \mathbb{R}^n$ for the configuration space of the system. For open chains all of whose joints are actuated, it is convenient and always possible to choose θ to be the vector of joint values. The generalized forces will then be denoted $\tau \in \mathbb{R}^n$. If θ_i is a revolute joint τ_i will correspond to a torque, while if θ_i is a prismatic joint τ_i will correspond to a force.

Once θ has been chosen and the generalized forces τ identified, the next step is to formulate the Lagrangian $\mathcal{L}(\theta, \dot{\theta})$ as

$$\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta), \tag{9.9}$$

where $\mathcal{K}(\theta, \dot{\theta})$ is the kinetic energy and $\mathcal{P}(\theta)$ the potential energy of the overall system. For rigid-link robots the kinetic energy can always be written in the form

$$\mathcal{K}(\theta) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}, \tag{9.10}$$

where $m_{ij}(\theta)$ is the (i, j) element of the $n \times n$ mass matrix $M(\theta)$; a constructive proof of this assertion is provided when we examine the Newton-Euler formulation in the next section. The dynamic equations are analytically obtained by evaluating the right-hand side of

$$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i}, \; i = 1, \ldots, n. \tag{9.11}$$

With the kinetic energy expressed as in Equation (9.10), the dynamics can be explicitly written as

$$\tau_i = \sum_{j=1}^{n} m_{ij}(\theta)\ddot{\theta}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_j\dot{\theta}_k + \frac{\partial \mathcal{P}}{\partial \theta_i}, \;\; i = 1, \ldots, n, \tag{9.12}$$

where the $\Gamma_{ijk}(\theta)$, known as the **Christoffel symbols of the first kind**, are defined as

$$\Gamma_{ijk}(\theta) = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i}\right). \tag{9.13}$$

The Lagrangian formulation has traditionally been regarded as the most direct way of obtaining a set of closed-form analytical equations for the dynamics. For open chain robots not only is this no longer true, but the formula for $\Gamma_{ijk}(\theta)$ above and our earlier examples offer a strong hint of how intractable the calculations can become, especially for robots with higher degrees of freedom. The Newton-Euler formulation on the other hand allows us to bypass the evaluation of these partial derivatives. However, as we show later, the Lagrangian formulation offers important insights into the structure of the dynamics equations, especially in the development of stable robot control laws.

9.2 Dynamics of a Single Rigid Body

9.2.1 Classical Formulation

Suppose a rigid body of mass $\mathfrak{m}$ has a reference frame {b} with axes $\{\hat{\mathrm{x}}_b, \hat{\mathrm{y}}_b, \hat{\mathrm{z}}_b\}$ attached to its center of mass. As the rigid body moves, the body frame also moves with linear velocity v and angular velocity w. Further assume that the rigid body is subject to an external force f. The external moment m generated by f with respect to the center of mass is then $\mathrm{m} = \mathrm{r} \times \mathrm{f}$, where r is the vector from the center of mass to the point on the body at which f is applied. Let h denote the angular momentum vector about the center of mass (we'll explain shortly how to calculate h). The dynamic equations for the rigid body are then given by

$$\mathrm{f} = \mathfrak{m}\frac{d}{dt}\mathrm{v} \tag{9.14}$$

$$\mathrm{m} = \frac{d}{dt}\mathrm{h}. \tag{9.15}$$

We now express these dynamic equations explicitly in frame {b} coordinates. First express the angular and linear velocity in {b}-frame coordinates by

$$\begin{aligned} \mathrm{w} &= \omega_x\,\hat{\mathrm{x}}_b + \omega_y\,\hat{\mathrm{y}}_b + \omega_z\,\hat{\mathrm{z}}_b \\ \mathrm{v} &= v_x\,\hat{\mathrm{x}}_b + v_y\,\hat{\mathrm{y}}_b + v_z\,\hat{\mathrm{z}}_b, \end{aligned}$$

with the column vectors $\omega_b = (\omega_x,\ \omega_y,\ \omega_z)^T$, $v_b = (v_x,\ v_y,\ v_z)^T$ accordingly defined. The linear acceleration a is then

$$\mathrm{a} = \frac{d}{dt}\mathrm{v} = (\dot{v}_x\,\hat{\mathrm{x}}_b + \dot{v}_y\,\hat{\mathrm{y}}_b + \dot{v}_z\,\hat{\mathrm{z}}_b) + v_x\,\dot{\hat{\mathrm{x}}}_b + v_y\,\dot{\hat{\mathrm{y}}}_b + v_z\,\dot{\hat{\mathrm{z}}}_b. \tag{9.16}$$

Substituting the rotating frame identities $\dot{\hat{\mathrm{x}}}_b = \mathrm{w}\times\hat{\mathrm{x}}_b$, $\dot{\hat{\mathrm{y}}}_b = \mathrm{w}\times\hat{\mathrm{y}}_b$, $\dot{\hat{\mathrm{z}}}_b = \mathrm{w}\times\hat{\mathrm{z}}_b$ (recall that these identities were used to extract the angular velocity vector $\omega_b \in \mathbb{R}^3$ from the rotation matrix $R(t)$ as $[\omega_b] = R^T\dot{R}$) into (9.16) leads to

$$\mathrm{a} = (\dot{v}_x\,\hat{\mathrm{x}}_b + \dot{v}_y\,\hat{\mathrm{y}}_b + \dot{v}_z\,\hat{\mathrm{z}}_b) + \mathrm{w}\times\mathrm{v}.$$

The {b}-frame vector representation $a_b \in \mathbb{R}^3$ of the linear acceleration is therefore

$$a_b = \dot{v}_b + (\omega_b \times v_b),$$

where $\dot{v}_b = (\dot{v}_x, \dot{v}_y, \dot{v}_z)^T$. Equation (9.14) expressed in {b}-frame coordinates now becomes

$$f_b = \mathfrak{m}(\dot{v}_b + \omega_b \times v_b). \tag{9.17}$$

We next express the angular momentum h in {b}-frame coordinates. When the {b} frame is attached to the body's center of mass as we have done here, the angular momentum assumes a particularly simple form. First, the 3×3 rotational inertia matrix of the rigid body is required; this can be obtained by imagining the rigid body as a collection of an infinite number of particles of mass $\mathfrak{m}_i$, each with coordinates $(x_i,\ y_i,\ z_i)$ with respect to the {b} frame. Denoting the rotational inertia matrix by $\mathcal{I}_b \in \mathbb{R}^{3\times 3}$, $\mathcal{I}_b$ is obtained via the following summation over all particles constituting the rigid body:

$$\begin{aligned} \mathcal{I}_b &= \begin{bmatrix} \sum \mathfrak{m}_i(y_i^2+z_i^2) & -\sum \mathfrak{m}_i x_i y_i & -\sum \mathfrak{m}_i x_i z_i \\ -\sum \mathfrak{m}_i x_i y_i & \sum \mathfrak{m}_i(x_i^2+z_i^2) & -\sum \mathfrak{m}_i y_i z_i \\ -\sum \mathfrak{m}_i x_i z_i & -\sum \mathfrak{m}_i y_i z_i & \sum \mathfrak{m}_i(x_i^2+y_i^2) \end{bmatrix} \\ &= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{xz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}. \end{aligned}$$

The matrix $\mathcal{I}_b$ defined in this way is constant, symmetric, and positive-definite. In the limit as the number of particles becomes infinite, the summations can be replaced by volume integrals over the body $\mathcal{B}$, with the particle masses m_i now replaced by a mass density function $\rho(x, y, z)$:

$$\begin{aligned}
\mathcal{I}_{xx} &= \int\int\int_{\mathcal{B}} (y^2 + z^2)\rho(x, y, z)\, dx\, dy\, dz \\
\mathcal{I}_{yy} &= \int\int\int_{\mathcal{B}} (x^2 + z^2)\rho(x, y, z)\, dx\, dy\, dz \\
\mathcal{I}_{zz} &= \int\int\int_{\mathcal{B}} (x^2 + y^2)\rho(x, y, z)\, dx\, dy\, dz \\
\mathcal{I}_{xy} &= \mathcal{I}_{yx} = -\int\int\int_{\mathcal{B}} xy\rho(x, y, z)\, dx\, dy\, dz \\
\mathcal{I}_{xz} &= \mathcal{I}_{zx} = -\int\int\int_{\mathcal{B}} xz\rho(x, y, z)\, dx\, dy\, dz \\
\mathcal{I}_{yz} &= \mathcal{I}_{zy} = -\int\int\int_{\mathcal{B}} yz\rho(x, y, z)\, dx\, dy\, dz.
\end{aligned}$$

If the body has a uniform mass density throughout, $\mathcal{I}_b$ is then determined exclusively by the shape of the rigid body. The calculation of $\mathcal{I}_b$ for some standard link shapes are covered in the exercises at the end of this chapter.

Express h In terms of the unit axes of the {b} frame as

$$\mathrm{h} = h_x\, \hat{\mathrm{x}}_b + h_y\, \hat{\mathrm{y}}_b + h_z\, \hat{\mathrm{z}}_b, \tag{9.18}$$

and define $h_b = (h_x,\ h_y,\ h_z)^T \in \mathbb{R}^3$. h_b is then obtained as

$$h_b = \mathcal{I}_b \omega_b. \tag{9.19}$$

Since Equation (9.15) calls for the derivative of h, differentiating (9.18) leads to

$$\frac{d}{dt}\mathrm{h} = (\dot{h}_x\, \hat{\mathrm{x}}_b + \dot{h}_y\, \hat{\mathrm{y}}_b + \dot{h}_z\, \hat{\mathrm{z}}_b) + \mathrm{w} \times \mathrm{h}.$$

The moment equation (9.15) expressed in {b} frame coordinates thus becomes

$$m_b = \mathcal{I}_b \dot{\omega}_b + \omega_b \times \mathcal{I}_b \omega_b. \tag{9.20}$$

where $m_b \in \mathbb{R}^3$ is the moment vector m in frame {b} coordinates. Equations (9.17) and (9.20) together constitute the dynamic equations of motion for the rigid body.

9.2.2 Twist-Wrench Formulation

Equations (9.17) and (9.20) can be written in the following combined form:

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & 0 \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}. \tag{9.21}$$

With the benefit of hindsight, and also making use of the fact that $[v]v = v \times v = 0$ and $[v]^T = -[v]$, we write (9.21) in the following equivalent form:

$$\begin{aligned} \begin{bmatrix} m_b \\ f_b \end{bmatrix} &= \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & [v_b] \\ 0 & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} - \begin{bmatrix} [\omega_b] & 0 \\ [v_b] & [\omega_b] \end{bmatrix}^T \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}. \end{aligned}$$

Written in this way, each of the terms can now be identified with six-dimensional spatial quantities as follows:

(i) (ω_b, v_b) and (m_b, f_b) can be respectively identified with the spatial velocity (or twist) $\mathcal{V}_b$ and spatial force (or wrench) $\mathcal{F}_b$:

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}, \quad \mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix}. \tag{9.22}$$

(ii) The **spatial inertia matrix** $\mathcal{G}_b \in \mathbb{R}^{6\times 6}$ is defined as follows:

$$\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix}, \tag{9.23}$$

where I denotes the 3×3 identity matrix. Note as an aside that the kinetic energy of the rigid body can be expressed in terms of the spatial inertia matrix as

$$\text{Kinetic Energy} = \frac{1}{2}\omega_b^T \mathcal{I}_b \omega_b + \frac{1}{2}\mathfrak{m} v_b^T v_b = \frac{1}{2}\mathcal{V}_b^T \mathcal{G}_b \mathcal{V}_b. \tag{9.24}$$

(iii) The **spatial momentum** $\mathcal{P}_b \in \mathbb{R}^6$ is defined as

$$\mathcal{P}_b = \begin{bmatrix} \mathcal{I}_b \omega_b \\ \mathfrak{m} v_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \mathcal{G}_b \mathcal{V}_b. \tag{9.25}$$

Observe that the $\mathcal{P}_b$ term in the dynamic equation is left-multiplied by the matrix

$$-\begin{bmatrix} [\omega_b] & 0 \\ [v_b] & [\omega_b] \end{bmatrix}^T. \tag{9.26}$$

We now explain the origin and geometric significance of this matrix. First, recall that the cross product of two vectors $\omega_1, \omega_2 \in \mathbb{R}^3$ can be calculated using skew-symmetric matrix notation as follows:

$$[\omega_1 \times \omega_2] = [\omega_1][\omega_2] - [\omega_2][\omega_1]. \tag{9.27}$$

The matrix in (9.26) can be thought of as a generalization of the cross product operation to six-dimensional twists. Specifically, given two twists $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$, we perform a calculation analogous to (9.27):

$$\begin{bmatrix} [\omega_1] & v_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\omega_1] & v_1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} [\omega_1][\omega_2] - [\omega_2][\omega_1] & [\omega_1]v_2 - [\omega_2]v_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega'] & v' \\ 0 & 0 \end{bmatrix},$$

which can be written more compactly in vector form as

$$\begin{bmatrix} \omega' \\ v' \end{bmatrix} = \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix} \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix}.$$

This generalization of the cross product to two twists $\mathcal{V}_1$ and $\mathcal{V}_2$ will be called the **Lie bracket** of $\mathcal{V}_1$ and $\mathcal{V}_2$.

Definition 9.1. Given two twists $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$, the **Lie bracket** of $\mathcal{V}_1$ and $\mathcal{V}_2$, denoted simultaneously by $[\mathcal{V}_1, \mathcal{V}_2]$ and $\mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2)$, is defined as follows:

$$[\mathcal{V}_1, \mathcal{V}_2] = \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix} \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2). \qquad (9.28)$$

Given $\mathcal{V} = (\omega, v)$, we further define the following notation for the 6×6 matrix representation $[\mathrm{ad}_{\mathcal{V}}]$:

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}. \qquad (9.29)$$

With this notation the Lie bracket $[\mathcal{V}_1, \mathcal{V}_2]$ can also be expressed as

$$[\mathcal{V}_1, \mathcal{V}_2] = \mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2) = [\mathrm{ad}_{\mathcal{V}_1}]\mathcal{V}_2. \qquad (9.30)$$

Definition 9.2. Given a twist $\mathcal{V} = (\omega, v)$ and wrench $\mathcal{F} = (m, f)$, define the mapping

$$\mathrm{ad}^T_{\mathcal{V}}(\mathcal{F}) = [\mathrm{ad}_{\mathcal{V}}]^T \mathcal{F} = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}^T \begin{bmatrix} m \\ f \end{bmatrix} = \begin{bmatrix} -[\omega]m - [v]f \\ -[\omega]f \end{bmatrix} \qquad (9.31)$$

Using the above notation and definitions the dynamic equations for a single rigid body can now be written as follows:

$$\begin{aligned} \mathcal{F}_b &= \mathcal{G}_b \dot{\mathcal{V}}_b - \mathrm{ad}^T_{\mathcal{V}_b}(\mathcal{P}_b) & (9.32) \\ &= \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^T \mathcal{G}_b \mathcal{V}_b. & (9.33) \end{aligned}$$

Note the similarity between (9.33) and the moment equation for a rotating rigid body:

$$m_b = \mathcal{I}_b \dot{\omega}_b - [\omega_b]^T \mathcal{I}_b \omega_b. \qquad (9.34)$$

Equation (9.34) is simply the rotational component of (9.33).

9.3 Inverse Dynamics of Open Chains

We now consider the inverse dynamics problem for an n-link open chain connected by one degree-of-freedom joints. Given the joint positions $\theta \in \mathbb{R}^n$, velocities $\dot{\theta} \in \mathbb{R}^n$, and accelerations $\ddot{\theta} \in \mathbb{R}^n$, the objective is to calculate the right-hand side of the dynamics equation

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}).$$

The main result will be a recursive inverse dynamics algorithm consisting of a forward and backward iteration stage. In the forward iteration, the velocities and accelerations of each link are propagated from the base to the tip, while in the backward iteration, the forces and moments experienced by each link are propagated from the tip to the base.

The following setting is assumed. A body-fixed reference frame $\{i\}$ is attached to the center of mass of each link i, $i = 1, \ldots, n$. The ground frame is denoted $\{0\}$, while the center of mass frame for the final link is denoted $\{n\}$. The displacement from frame $\{i-1\}$ to $\{i\}$, denoted $T_{i-1,i} \in SE(3)$, is expressed in the following form:

$$T_{i-1,i} = M_{i-1,i} e^{[\mathcal{A}_i]\theta_i}, \tag{9.35}$$

where $M_{i-1,i} \in SE(3)$, and $A_i = (\omega_i, v_i)$ is the twist vector for joint i (assuming θ_i is set to zero) expressed in frame $\{i-1\}$ coordinates. If the forward kinematics is expressed in the product-of-exponentials form

$$T_{0n} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M, \tag{9.36}$$

then the forward kinematics up to each link frame $\{i\}$ can be written

$$T_{0i} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_i]\theta_i} M_i \tag{9.37}$$

$$M_i = M_{01} M_{12} \cdots M_{i-1,i} \tag{9.38}$$

for $i = 1, \ldots, n$, where $M_i \in SE(3)$ denotes the configuration of link frame $\{i\}$ in the zero position. For each $i = 1, \ldots, n$, the following can be established via direct calculation:

$$M_{i-1,i} = M_{i-1}^{-1} M_i \tag{9.39}$$

$$\mathcal{A}_i = \mathrm{Ad}_{M_i^{-1}}(\mathcal{S}_i). \tag{9.40}$$

We further define the following notation:

(i) Denote the velocity of link frame $\{i\}$ expressed in frame $\{i\}$ coordinates by $\mathcal{V}_i = (\omega_i, v_i)$. Note that $\mathcal{V}_i$ is obtained from $[\mathcal{V}_i] = T_{0i}^{-1}\dot{T}_{0i}$.

(ii) Let $\mathcal{G}_i \in \mathbb{R}^{6\times 6}$ denote the 6×6 inertia matrix of link i, expressed relative to link frame $\{i\}$. Since we assume here that all link frames are situated at the link center of mass, $\mathcal{G}_i$ has the block-diagonal form

$$\mathcal{G}_i = \begin{bmatrix} \mathcal{I}_i & 0 \\ 0 & \mathfrak{m}_i I \end{bmatrix}, \tag{9.41}$$

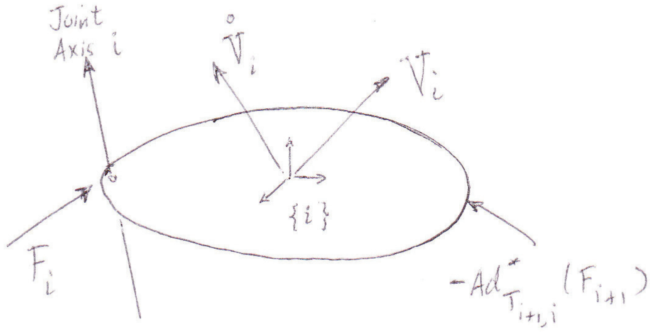

Figure 9.1: Free body diagram illustrating the moments and forces exerted on link i.

where $\mathcal{I}_i$ denotes the 3×3 rotational inertia matrix of link i, and $\mathfrak{m}_i$ is the link mass.

(iii) Denote by $\mathcal{F}_i = (m_i, f_i)$ the spatial force transmitted from link $i-1$ to link i, expressed in frame $\{i\}$ coordinates. Note that $\mathcal{F}_i$ is transmitted entirely through joint i, since this is the only point of contact between link $i-1$ and link i.

With the above notation and definitions, we now consider the free-body diagram for link i as shown in Figure 9.1. Note that $\mathcal{F}_{i+1}$ is the wrench applied by link i to link $i+1$, expressed in frame $\{i+1\}$ coordinates. What is needed is the wrench applied by link $i+1$ to link i, expressed in frame $\{i\}$ coordinates; using the wrench transformation rule under a change of reference frames, this is given by

$$\mathrm{Ad}^T_{T_{i+1,i}}(-\mathcal{F}_{i+1}) = -\mathrm{Ad}^T_{T_{i+1,i}}(\mathcal{F}_{i+1}).$$

The equations of motion for link i can therefore be written

$$\mathcal{G}_i \dot{\mathcal{V}}_i = \mathrm{ad}^T_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i) + \mathcal{F}_i - \mathrm{Ad}^T_{T_{i+1,i}}(\mathcal{F}_{i+1}). \tag{9.42}$$

The joint torque $\tau_i \in \mathbb{R}$ at joint i is then the projection of the wrench $\mathcal{F}_i$ onto the joint twist $\mathcal{A}_i$:

$$\tau_i = \mathcal{F}_i^T \mathcal{A}_i. \tag{9.43}$$

We now derive the forward iteration of link velocities and accelerations from the base to the tip. First note that

$$[\mathcal{V}_1] = T_{01}^{-1} \dot{T}_{01} = [\mathcal{A}_1 \dot{\theta}_1] \tag{9.44}$$

and

$$\begin{aligned} [\mathcal{V}_2] &= T_{02}^{-1} \dot{T}_{02} \\ &= T_{12}^{-1}(T_{01}^{-1}\dot{T}_{01}) T_{12} + T_{12}^{-1} \dot{T}_{12} \\ &= T_{12}^{-1} [\mathcal{V}_1] T_{12} + [\mathcal{A}_2 \dot{\theta}_2], \end{aligned} \tag{9.45}$$

or equivalently, $\mathcal{V}_2 = \mathrm{Ad}_{T_{21}}(\mathcal{V}_1) + \mathcal{A}_2\dot{\theta}_2$. Repeating this procedure for the subsequent links, it can be established that

$$\mathcal{V}_i = \mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \mathcal{A}_i\dot{\theta}_i, \quad i = 1, \ldots, n. \tag{9.46}$$

The accelerations $\dot{\mathcal{V}}_i$ can also be found recursively. Noting that

$$[\dot{\mathcal{V}}_i] = \frac{d}{dt}T_{i-1,i}[\mathcal{V}_i]T_{i-1,i}^{-1} + T_{i-1,i}[\dot{\mathcal{V}}_i]T_{i-1,i}^{-1} + T_{i-1,i}[\mathcal{V}_i]\frac{d}{dt}T_{i-1,i}^{-1} + [\mathcal{A}_i]\ddot{\theta}_i,$$

and

$$\begin{aligned} \frac{d}{dt}T_{i-1,i} &= M_{i-1,i}[\mathcal{A}_i]e^{[\mathcal{A}_i]\theta_i}\dot{\theta}_i = M_{i-1,i}e^{[\mathcal{A}_i]\theta_i}[\mathcal{A}_i]\dot{\theta}_i \\ \frac{d}{dt}T_{i-1,i}^{-1} &= -T_{i-1,i}^{-1}\dot{T}_{i-1,i}T_{i-1,i}^{-1}, \end{aligned}$$

it can be shown that

$$\dot{\mathcal{V}}_i = \mathcal{A}_i\ddot{\theta}_i + \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + [\mathrm{Ad}_{T_{i-1,i}}(\mathcal{V}_{i-1}), \mathcal{A}_i\dot{\theta}_i], \tag{9.47}$$

where $\left[\mathrm{Ad}_{T_{i-1,i}}(\mathcal{V}_{i-1}), \mathcal{A}_i\right]$ denotes the Lie bracket of $\mathrm{Ad}_{T_{i-1,i}}(\mathcal{V}_{i-1})$ with $\mathcal{A}_i$. Note that since $[\mathcal{A}_i, \mathcal{A}_i] = 0$ and $\mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) = \mathcal{V}_i - \mathcal{A}_i\dot{\theta}_i$, one obtains the alternative but equivalent formula

$$\dot{\mathcal{V}}_i = \mathcal{A}_i\ddot{\theta}_i + \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + [\mathcal{V}_i, \mathcal{A}_i\dot{\theta}_i]. \tag{9.48}$$

The above formulas for the velocities and accelerations, together with the dynamic equations for any given link, can now be organized into a two-stage forward-backward iterative algorithm for the inverse dynamics. Before doing so, we examine how to include gravity in the dynamics. One way to simulate the effects of gravity is to set the base frame to have an acceleration $-\mathfrak{g}$, where $\mathfrak{g} \in \mathbb{R}^3$ denotes the gravitational acceleration vector as expressed in base frame coordinates. In this case it is important to remember that the link acceleration calculated from the recursive algorithm is not its true acceleration, but rather its true acceleration minus $\mathfrak{g}$.

The algorithm is initialized by providing initial values for $\mathcal{V}_0$, $\dot{\mathcal{V}}_0$, and $\mathcal{F}_{\text{tip}}$, where $\mathcal{V}_0$ and $\dot{\mathcal{V}}_0$ are respectively the spatial velocity and spatial acceleration of the base frame expressed in base frame coordinates, and $\mathcal{F}_{\text{tip}}$ is the external spatial force applied to some point on the final link, expressed in end-effector frame coordinates. The joint trajectory $\theta(t)$ and its derivatives $\dot{\theta}$, $\ddot{\theta}$ are also assumed given as input.

Newton-Euler Inverse Dynamics Algorithm

- **Preliminaries**: Each link frame $\{i\}$ is assumed attached to the link's center of mass. The forward kinematics from the base frame to link frame $\{i\}$ is of the form

$$T_{0i} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_i]\theta_i} M_i, \; i = 1, \ldots, n. \tag{9.49}$$

Define $M_{i-1,i}$ to be the displacement from link frame $\{i-1\}$ to link frame $\{i\}$. Then $M_i = M_{01}M_{12} \cdots M_{i-1,i}$ and $M_{i-1,i} = M_{i-1}^{-1} M_i$, $i = 1, \ldots, n$. The displacement between link frames $\{i-1\}$ and $\{i\}$ is

$$T_{i-1,i} = M_{i-1,i} e^{[\mathcal{A}_i]\theta_i}, \tag{9.50}$$

where

$$\mathcal{A}_i = \mathrm{Ad}_{M_i^{-1}}(\mathcal{S}_i), \quad i = 1, \ldots, n. \tag{9.51}$$

With respect to the link frame attached at its center of mass, the 6×6 spatial inertia $\mathcal{G}_i$ of link i is defined as

$$\mathcal{G}_i = \begin{bmatrix} \mathcal{I}_i & 0 \\ 0 & \mathrm{m}_i I \end{bmatrix}, \tag{9.52}$$

where $\mathcal{I}_i \in \mathbb{R}^{3\times 3}$ is its rotational inertia matrix, and m_i is the mass of link i. Define the twist $\mathcal{V}_0 = (\omega_0, v_0)$ to be the spatial velocity of the base frame, expressed in base frame coordinates. Define $\mathfrak{g} \in \mathbb{R}^3$ to be the gravity vector expressed in base frame $\{0\}$ coordinates. Define $\mathcal{F}_{\mathrm{tip}} = (m_{\mathrm{tip}}, f_{\mathrm{tip}})$ to be the wrench applied to some point on link n, expressed in end-effector frame coordinates.

- **Initialization**: $\mathcal{V}_0 = \text{given}, \; \dot{\mathcal{V}}_0 = (0, \mathfrak{g}), \; \mathcal{F}_{n+1} = \mathcal{F}_{\mathrm{tip}}$.

- **Forward Iteration**: for $i = 1$ to n do

$$T_{i-1,i} = M_{i-1,i} e^{[\mathcal{A}_i]\theta_i} \tag{9.53}$$
$$\mathcal{V}_i = \mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \mathcal{A}_i \dot{\theta}_i \tag{9.54}$$
$$\dot{\mathcal{V}}_i = \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + [\mathcal{V}_i, \mathcal{A}_i]\dot{\theta}_i + \mathcal{A}_i \ddot{\theta}_i \tag{9.55}$$

- **Backward Iteration**: for $i = n$ to 1 do

$$\mathcal{F}_i = \mathrm{Ad}^T_{T_{i+1,i}}(\mathcal{F}_{i+1}) + \mathcal{G}_i \dot{\mathcal{V}}_i - \mathrm{ad}^T_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i) \tag{9.56}$$
$$\tau_i = \mathcal{F}_i^T \mathcal{A}_i. \tag{9.57}$$

As noted earlier, the recursion formula Equation (9.55) for the acceleration $\dot{\mathcal{V}}_i$ can also be replaced by the equivalent formula

$$\dot{\mathcal{V}}_i = \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + [\mathrm{Ad}_{T_{i-1,i}}(\mathcal{V}_{i-1}), \mathcal{A}_i]\dot{\theta}_i + \mathcal{A}_i \ddot{\theta}_i. \tag{9.58}$$

9.4 Dynamic Equations in Closed Form

In this section we show how the equations in the recursive inverse dynamics algorithm can be organized into a closed-form set of dynamics equations of the form $\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$.

Before doing so, we prove our earlier assertion that the total kinetic energy $\mathcal{K}$ of the robot can be expressed as $\mathcal{K} = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$. We do so by noting that $\mathcal{K}$ can be expressed as the sum of the kinetic energies of each link:

$$\mathcal{K} = \frac{1}{2}\sum_{i=1}^{n} \mathcal{V}_i^T \mathcal{G}_i \mathcal{V}_i, \tag{9.59}$$

where $\mathcal{V}_i$ is the spatial velocity of link frame $\{i\}$, and $\mathcal{G}_i$ is the spatial inertia matrix of link i as defined by Equation (9.52) (both are expressed in link frame $\{i\}$ coordinates). Let $T_{0i}(\theta_1, \ldots, \theta_i)$ denote the forward kinematics from the base frame $\{0\}$ to link frame $\{i\}$, and let $J_{ib}(\theta)$ denote the body Jacobian obtained from $T_{0i}^{-1}\dot{T}_{0i}$. Note that J_{ib} as defined is a $6 \times i$ matrix; we turn it into a $6 \times n$ matrix by filling in all entries of the last $n - i$ columns with zeros. With this definition of J_{bi}, we can write

$$\mathcal{V}_i = J_{ib}(\theta)\dot{\theta}, \quad i = 1, \ldots, n.$$

The kinetic energy can then be written

$$\mathcal{K} = \frac{1}{2}\dot{\theta}^T \left(\sum_{i=1}^{n} J_{ib}(\theta)^T \mathcal{G}_i J_{ib}(\theta) \right) \dot{\theta}. \tag{9.60}$$

The term inside the parentheses is precisely the mass matrix $M(\theta)$:

$$M(\theta) = \sum_{i=1}^{n} J_{ib}(\theta)^T \mathcal{G}_i J_{ib}(\theta). \tag{9.61}$$

Some of the exericses at the end of this chapter examine ways to recursively compute the entries of $M(\theta)$.

We now return to the original task of deriving a closed-form set of dynamic equations. We start by defining the following stacked vectors:

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_n \end{bmatrix} \in \mathbb{R}^{6n} \tag{9.62}$$

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_1 \\ \vdots \\ \mathcal{F}_n \end{bmatrix} \in \mathbb{R}^{6n}. \tag{9.63}$$

Further define the following matrices:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{A}_n \end{bmatrix} \in \mathbb{R}^{6n\times n} \tag{9.64}$$

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{G}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathcal{G}_n \end{bmatrix} \in \mathbb{R}^{6n\times 6n} \tag{9.65}$$

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{V}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{V}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{V}_n}] \end{bmatrix} \in \mathbb{R}^{6n\times 6n} \tag{9.66}$$

$$\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right] = \begin{bmatrix} [\mathrm{ad}_{\mathcal{A}_1\dot{\theta}_1}] & 0 & \cdots & 0 \\ 0 & [\mathrm{ad}_{\mathcal{A}_2\dot{\theta}_2}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [\mathrm{ad}_{\mathcal{A}_n\dot{\theta}_n}] \end{bmatrix} \in \mathbb{R}^{6n\times 6n} \tag{9.67}$$

$$\Gamma(\theta) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ [\mathrm{Ad}_{T_{21}}] & 0 & \cdots & 0 & 0 \\ 0 & [\mathrm{Ad}_{T_{32}}] & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \left[\mathrm{Ad}_{T_{n,n-1}}\right] & 0 \end{bmatrix} \in \mathbb{R}^{6n\times 6n}. \tag{9.68}$$

We write $\Gamma(\theta)$ to emphasize the dependence of Γ on θ. Finally, define the following stacked vectors:

$$\mathcal{V}_{\mathrm{base}} = \begin{bmatrix} \mathrm{Ad}_{T_{10}}(\mathcal{V}_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \tag{9.69}$$

$$\dot{\mathcal{V}}_{\mathrm{base}} = \begin{bmatrix} \mathrm{Ad}_{T_{10}}(\dot{\mathcal{V}}_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n} \tag{9.70}$$

$$\mathcal{F}_{\mathrm{tip}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ad}^T_{T_{n+1,n}}(\mathcal{F}_{n+1}) \end{bmatrix} \in \mathbb{R}^{6n}. \tag{9.71}$$

Note that $\mathcal{A} \in \mathbb{R}^{6n\times n}$ and $\mathcal{G} \in \mathbb{R}^{6n\times 6n}$ are constant block-diagonal matrices, in which $\mathcal{A}$ contains only the kinematic parameters, while $\mathcal{G}$ contains only the mass and inertial parameters for each link.

With the above definitions, our earlier recursive inverse dynamics algorithm can be assembled into the following set of matrix equations:

$$\begin{aligned}
\mathcal{V} &= \Gamma(\theta)\mathcal{V} + \mathcal{A}\dot{\theta} + \mathcal{V}_{\text{base}} &(9.72)\\
\dot{\mathcal{V}} &= \Gamma(\theta)\dot{\mathcal{V}} + \mathcal{A}\ddot{\theta} + [\text{ad}_{\mathcal{A}\dot{\theta}}](\Gamma(\theta)\mathcal{V} + \mathcal{V}_{\text{base}}) + \dot{\mathcal{V}}_{\text{base}} &(9.73)\\
\mathcal{F} &= \Gamma^T(\theta)\mathcal{F} + \mathcal{G}\dot{\mathcal{V}} - [\text{ad}_{\mathcal{V}}]^T\mathcal{G}\mathcal{V} + \mathcal{F}_{\text{tip}} &(9.74)\\
\tau &= \mathcal{A}^T\mathcal{F}. &(9.75)
\end{aligned}$$

$\Gamma(\theta)$ has the property that $\Gamma^n(\theta) = 0$ (such a matrix is said to be nilpotent of order n), and one consequence verifiable through direct calculation is that $(I - \Gamma(\theta))^{-1} = I + \Gamma(\theta) + \ldots + \Gamma^{n-1}(\theta)$. Defining $\mathcal{L}(\theta) = (I - \Gamma(\theta))^{-1}$, it can further be verified via direct calculation that

$$\mathcal{L}(\theta) = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ [\text{Ad}_{T_{21}}] & I & 0 & \cdots & 0 \\ [\text{Ad}_{T_{31}}] & [\text{Ad}_{T_{32}}] & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [\text{Ad}_{T_{n1}}] & [\text{Ad}_{T_{n2}}] & [\text{Ad}_{T_{n3}}] & \cdots & I \end{bmatrix} \in \mathbb{R}^{6n\times 6n}. \tag{9.76}$$

We write $\mathcal{L}(\theta)$ to emphasize the dependence of $\mathcal{L}$ on θ. The earlier matrix equations can now be reorganized as

$$\begin{aligned}
\mathcal{V} &= \mathcal{L}(\theta)\left(\mathcal{A}\dot{\theta} + \mathcal{V}_{\text{base}}\right) &(9.77)\\
\dot{\mathcal{V}} &= \mathcal{L}(\theta)\left(\mathcal{A}\ddot{\theta} + [\text{ad}_{\mathcal{A}\dot{\theta}}]\Gamma(\theta)\mathcal{V} + [\text{ad}_{\mathcal{A}\dot{\theta}}]\mathcal{V}_{\text{base}} + \dot{\mathcal{V}}_{\text{base}}\right) &(9.78)\\
\mathcal{F} &= \mathcal{L}^T(\theta)\left(\mathcal{G}\dot{\mathcal{V}} - [\text{ad}_{\mathcal{V}}]^T\mathcal{G}\mathcal{V} + \mathcal{F}_{\text{tip}}\right) &(9.79)\\
\tau &= \mathcal{A}^T\mathcal{F}. &(9.80)
\end{aligned}$$

If an external wrench $\mathcal{F}_{\text{tip}}$ is applied at the tip, this can be included into the following dynamics equation:

$$\tau = M(\theta)\ddot{\theta} + c(\theta,\dot{\theta}) + g(\theta) + J^T(\theta)\mathcal{F}_{\text{tip}}, \tag{9.81}$$

where $J(\theta)$ denotes the Jacobian of the forward kinematics expressed in the same reference frame as $\mathcal{F}_{\text{tip}}$, and

$$\begin{aligned}
M(\theta) &= \mathcal{A}^T\mathcal{L}^T(\theta)\mathcal{G}\mathcal{L}(\theta)\mathcal{A} &(9.82)\\
c(\theta,\dot{\theta}) &= \mathcal{A}^T\mathcal{L}^T(\theta)\left(\mathcal{G}\mathcal{L}(\theta)\,[\text{ad}_{\mathcal{A}\dot{\theta}}]\Gamma(\theta) - [\text{ad}_{\mathcal{V}}]^T\mathcal{G}\right)\mathcal{L}(\theta)\mathcal{A}\dot{\theta} &(9.83)\\
g(\theta) &= \mathcal{A}^T\mathcal{L}^T(\theta)\mathcal{G}\mathcal{L}(\theta)\dot{\mathcal{V}}_{\text{base}}. &(9.84)
\end{aligned}$$

The $g(\theta)$ term reflects gravitational forces, while $c(\theta,\dot{\theta})$ represents the Coriolis and centrifugal forces. Comparing these equations with the Lagrangian form of the dynamics, i.e.,

$$\tau_i = \sum_{j=1}^{n} m_{ij}(\theta)\ddot{\theta}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_j\dot{\theta}_k + \frac{\partial \mathcal{P}}{\partial \theta_i}, \quad i = 1, \ldots, n, \tag{9.85}$$

where the $\Gamma_{ijk}(\theta)$ are

$$\Gamma_{ijk}(\theta) = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i}\right), \tag{9.86}$$

we can see that elements of the $c(\theta, \dot{\theta})$ term can be identified with

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_j\dot{\theta}_k$$

and are thus quadratic in the $\dot{\theta}_i$. Elements of the gravity term $g(\theta)$ can be identified with $\frac{\partial \mathcal{P}}{\partial \theta_i}$. With the Newton-Euler formulation, the partial derivative terms appearing in $\Gamma_{ijk}(\theta)$ can be evaluated directly from (9.83) without taking derivatives. Further, by defining the matrix $C(\theta, \dot{\theta}) \in \mathbb{R}^{n\times n}$ as

$$c_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_k = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i}\right)\dot{\theta}_k, \tag{9.87}$$

where c_{ij} denotes the (i, j) entry of $C(\theta, \dot{\theta})$, it can be seen that $c(\theta, \dot{\theta})$ can be expressed as

$$c(\theta, \dot{\theta}) = C(\theta, \dot{\theta})\dot{\theta}. \tag{9.88}$$

The matrix $C(\theta, \dot{\theta})$ is called the **Coriolis matrix**. The following property, referred to as the **passivity property**, turns out to have important ramifications in proving the stability of certain robot control laws.

Proposition 9.1. *The matrix $\dot{M}(\theta) - 2C(\theta, \dot{\theta}) \in \mathbb{R}^{n\times n}$, where $M(\theta) \in \mathbb{R}^{n\times n}$ is the mass matrix and dot$M(\theta)$ its time derivative, and $C(\theta, \dot{\theta}) \in \mathbb{R}^{n\times n}$ is the Coriolis matrix as defined in (9.87), is skew-symmetric.*

Proof. The (i, j) component of $\dot{M} - 2C$ is

$$\begin{aligned}
\dot{m}_{ij}(\theta) - 2c_{ij}(\theta, \dot{\theta}) &= \sum_{k=1}^{n} \frac{\partial m_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial m_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial m_{ik}}{\partial \theta_j}\dot{\theta}_k + \frac{\partial m_{kj}}{\partial \theta_i}\dot{\theta}_k \\
&= \sum_{k=1}^{n} \frac{\partial m_{kj}}{\partial \theta_i}\dot{\theta}_k - \frac{\partial m_{ik}}{\partial \theta_j}\dot{\theta}_k.
\end{aligned}$$

By switching the indices i and j, it can be seen that

$$\dot{m}_{ji}(\theta) - 2c_{ji}(\theta, \dot{\theta}) = -(\dot{m}_{ij}(\theta) - 2c_{ij}(\theta, \dot{\theta})),$$

thus proving that $(\dot{M} - 2C)^T = -(\dot{M} - 2C)$ as claimed. □

The passivity property will be used later in the chapter on robot control.

9.5 Forward Dynamics of Open Chains

We now consider the forward dynamics problem, where a torque trajectory $\tau(t)$ together with a set of initial conditions on θ and $\dot{\theta}$ is assumed given, and the objective is to integrate the dynamic equations $\tau(t) = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta})$ to obtain the joint trajectory $\theta(t)$. The simplest numerical scheme for integrating the general first-order differential equation $\dot{q} = f(q, t)$, $q \in \mathbb{R}^n$, is via the Euler iteration

$$q(t + h) = q(t) + hf(q(t), t),$$

where the positive scalar h denotes the timestep. The dynamic equations can be converted to a first-order differential equation by taking advantage of the fact that $M(\theta)$ is always invertible: setting $q_1 = \theta$, $q_2 = \dot{\theta}$, and $q = (q_1, q_2) \in \mathbb{R}^{2n}$, we can write

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= M^{-1}(q_1)\left(\tau(t) - b(q_1, q_2)\right), \end{aligned}$$

which is of the form $\dot{q} = f(q, t)$. The Euler integration scheme for this equation is thus of the form

$$\begin{aligned} q_1(t + h) &= q_1(t) + hq_2(t) \\ q_2(t + h) &= q_2(t) + h\left(M(q_1(t))^{-1}\left(\tau(t) - b(q_1(t), q_2(t))\right)\right). \end{aligned}$$

Given a set of initial values for $q_1(0) = \theta(0)$ and $q_2(0) = \dot{\theta}(0)$, the above equations can then be iterated forward in time to numerically obtain the motion $\theta(t) = q_1(t)$.

Note that the above iteration appears to require the evaluation of $M^{-1}(\theta)$, which can be computationally expensive. In fact, it is possible to integrate these equations without having to explicitly compute the inverse of $M(\theta)$. The closed-form dynamic equations can be arranged as

$$M(\theta)\ddot{\theta} = \tau(t) - b(\theta, \dot{\theta}). \tag{9.89}$$

Setting $\ddot{\theta}$ to zero in (9.89) leads to $\tau = b(\theta, \dot{\theta})$. Therefore by running the inverse dynamics algorithm with $\ddot{\theta}(t)$ set to zero and $(\theta(t), \dot{\theta}(t))$ set to their current values, $b(\theta(t), \dot{\theta}(t))$ can now be determined. Subtracting this from the given value of $\tau(t)$ then results in the right-hand side of (9.89). Therefore, with a means of evaluating $M(\theta)$, it is a straightforward matter to obtain $\ddot{\theta}(t)$ as the solution to the linear equation $Ax = c$, where $A = M(\theta(t)) \in \mathbb{R}^{n \times n}$ is assured to be nonsingular, and $c = \tau(t) - b(\theta(t), \dot{\theta}(t))$ is known.

Once a numerical integration scheme has been chosen, solving the forward dynamics then reduces to a procedure for evaluating $\ddot{\theta}$ from given values for θ, $\dot{\theta}$, and τ. In the following algorithm we allow for the possibility of an external spatial force $\mathcal{F}_{\text{tip}}$ applied to the final link.

Algorithm for Calculating the Joint Acceleration: *GetJointAccel*(θ, $\dot{\theta}$, τ, τ_{ext})

- **Prerequisites**: Algorithms for calculating the inverse dynamics, and the mass matrix, are assumed available. An algorithm for solving the linear system $Ax = c$ for $x \in \mathbb{R}^n$, with given $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ nonsingular, is also assumed available.
- **Inputs**: Current values for θ, $\dot{\theta}$, the input torque τ. If an external spatial tip force $\mathcal{F}_{\text{tip}}$ is also given, it is transformed via the static force-torque relation to $\tau_{\text{ext}} = J^T \mathcal{F}_{\text{tip}}$, where the Jacobian $J(\theta)$ is expressed in terms of the same reference frame as $\mathcal{F}_{\text{tip}}$.
- **Output**: The joint acceleration $\ddot{\theta}$.
- **Initialization**: Assign temporary storage variables $A \in \mathbb{R}^{n\times n}$, $\gamma \in \mathbb{R}^n$, $c \in \mathbb{R}^n$.
- **Inverse dynamics calculation**: Calculate the inverse dynamics with θ and $\dot{\theta}$ set to their given values, and $\ddot{\theta}$ set to zero; store the output joint torques in γ, and set $c = \tau - \gamma - \tau_{\text{ext}}$.
- **Evaluation of mass matrix**: Calculate the mass matrix for the given θ, and store the result in A.
- **Calculation of joint acceleration**: Solve the linear system $Ax = c$ for x; the resulting joint acceleration $\ddot{\theta}$ is then given by x.

With the above algorithm for calculating the joint acceleration, various numerical schemes for integrating the forward dynamics can be implemented; here we present an algorithm for the most basic Euler method described above:

Euler Integration Algorithm for Forward Dynamics

- **Prerequisites**: Function *GetJointAccel*$(\theta, \dot{\theta}, \tau, \tau_{\text{ext}})$ required.
- **Inputs**: Initial conditions $\theta(0)$ and $\dot{\theta}(0)$, input torques $\tau(t)$ and τ_{ext} for $t \in [0, t_f]$, integration timestep $h > 0$.
- **Output**: Joint trajectory values $\theta[k] = \theta(hk)$, $k = 0, \ldots, N$.
- **Initialization**: Set $N = t_f/h$.
- **Iteration**: For $k = 1$ to N do

$$\begin{aligned} \ddot{\theta}[k] &= \mathit{GetJointAccel}(\theta[k], \dot{\theta}[k], \tau[k], \tau_{\text{ext}}[k]); \\ \theta[k+1] &= \theta[k] + h\dot{\theta}[k]; \\ \dot{[\theta]}(k+1) &= \dot{\theta}[k] + h\ddot{\theta}[k]; \end{aligned}$$

- **Joint trajectory**: $\theta[k] = \theta(hk)$, $k = 0, \ldots, N$.

9.6 Dynamics in Task Space Coordinates

In this section we consider how the dynamic equations change under a transformation to coordinates of the end-effector frame (task space coordinates). To keep things simple we consider a six degree-of-freedom open chain with joint space dynamics

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}), \ \ \theta \in \mathbb{R}^6, \ \tau \in \mathbb{R}^6. \tag{9.90}$$

We also ignore for the time being any external spatial forces that may be applied. The spatial velocity $\mathcal{V} = (\omega, v)$ of the end-effector is related to the joint velocity $\dot{\theta}$ by

$$\mathcal{V} = J(\theta)\dot{\theta}, \tag{9.91}$$

with the understanding that $\mathcal{V}$ and $J(\theta)$ are always expressed in terms of the same reference frame. The time derivative $\dot{\mathcal{V}}$ is then

$$\dot{\mathcal{V}} = \dot{J}(\theta)\dot{\theta} + J(\theta)\ddot{\theta}. \tag{9.92}$$

At configurations θ where $J(\theta)$ is invertible, we have

$$\dot{\theta} = J^{-1}\mathcal{V} \tag{9.93}$$
$$\ddot{\theta} = J^{-1}\dot{\mathcal{V}} - J^{-1}\dot{J}J^{-1}\mathcal{V}. \tag{9.94}$$

The second term in (9.94) follows from the general matrix identity $\frac{d}{dt}(A^{-1}A) = \frac{d}{dt}A^{-1} \cdot A + A^{-1} \cdot \frac{d}{dt}A$ for any invertible and differentiable matrix $A(t)$. Substituting for $\dot{\theta}$ and $\ddot{\theta}$ in Equation (9.90) leads to

$$\tau = M\left(J^{-1}\dot{\mathcal{V}} - J^{-1}\dot{J}J^{-1}\mathcal{V}\right) + J^{-T}b(\theta, \dot{\theta}), \tag{9.95}$$

where J^{-T} denotes $(J^{-1})^T = (J^T)^{-1}$. Pre-multiply both sides by J^{-T} to get

$$\begin{aligned} J^{-T}\tau \;=\; & J^{-T}MJ^{-1}\dot{\mathcal{V}} - J^{-T}MJ^{-1}\dot{J}J^{-1}\mathcal{V} \\ & + J^{-T}b(\theta, J^{-1}\mathcal{V}). \end{aligned} \tag{9.96}$$

Expressing $J^{-T}\tau$ as the spatial force $\mathcal{F}$, the above can be written

$$\mathcal{F} = \Lambda(\theta)\dot{\mathcal{V}} + \eta(\theta, \mathcal{V}), \tag{9.97}$$

where

$$\Lambda(\theta) = J^{-T}M(\theta)J^{-1} \tag{9.98}$$
$$\eta(\theta, \mathcal{V}) = J^{-T}b(\theta, J^{-1}\mathcal{V}) - \Lambda\dot{J}J^{-1}\mathcal{V}. \tag{9.99}$$

These are the dynamic equations expressed in end-effector frame coordinates. If an external spatial force $\mathcal{F}$ is applied to the end-effector frame, then assuming zero joint torques, the motion of the end-effector frame is governed by these equations. Note the dependence of $\Lambda(\theta)$ and $\eta(\theta, \mathcal{V})$ on θ. If θ were replaced by its inverse kinematics solution $\theta = T^{-1}(X)$, then one would obtain a differential equation strictly in terms of the end-effector frame's displacement $X \in SE(3)$ and spatial velocity $\mathcal{V}$. In practice, since X is usually obtained by measuring θ and substituting into the forward kinematics, it is preferable to leave the dependence on θ explicit.