Chapter 6

Velocity Kinematics and Statics

A robot's **velocity kinematics** refers to the relationship between the robot's joint rates and the linear and angular velocity of its end-effector. Whereas the forward kinematics of typical robots is usually nonlinear and complicated, it turns out that the equations for velocity kinematics are linear: at any instant during a robot's motion, the end-effector's linear and angular velocity can be obtained simply by multiplying the joint rate vector by a (configuration-dependent) **Jacobian** matrix. This linear relationship is exploited to great effect in many applications, ranging from algorithms for inverse kinematics and trajectory generation to manipulation planning and control. The Jacobian matrix also turns out to play a central role in tasks involving static and dynamic contact between the end-effector and the environment.

In this chapter we derive the Jacobian matrix for open chains, and examine its role in velocity analysis, statics, and the identification of kinematic singularities. Later chapters on inverse kinematics, motion planning, and control will also draw upon these concepts in a fundamental way. The material in this chapter is based upon the treatment of rigid body velocities given in Chapter 3, and it may be useful to review this material first.

6.1 Manipulator Jacobian

6.1.1 Space Jacobian

In this section we derive the relationship between an open chain's joint rate vector $\dot{\theta}$ and the end-effector's spatial velocity $\mathcal{V}_s$. We first recall a few basic properties from linear algebra and linear differential equations: (i) if $A, B \in \mathbb{R}^{n \times n}$ are both invertible, then $(AB)^{-1} = B^{-1}A^{-1}$; (ii) if $A \in \mathbb{R}^{n \times n}$ is constant and $\theta(t)$ is a scalar function of t, then $\frac{d}{dt}e^{A\theta} = Ae^{A\theta}\dot{\theta} = e^{A\theta}A\dot{\theta}$; (iii) $(e^{A\theta})^{-1} = e^{-A\theta}$.

Now consider an n-link open chain whose forward kinematics is expressed in the following product of exponentials form:

$$T(\theta_1,\ldots,\theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M. \tag{6.1}$$

The spatial velocity of the end-effector frame with respect to the fixed frame, $\mathcal{V}_s$, is given by $[\mathcal{V}_s] = \dot{T}T^{-1}$, where

$$\begin{aligned} \dot{T} &= (\frac{d}{dt}e^{[\mathcal{S}_1]\theta_1})\cdots e^{[\mathcal{S}_n]\theta_n}M + e^{[\mathcal{S}_1]\theta_1}(\frac{d}{dt}e^{[\mathcal{S}_2]\theta_2})\cdots e^{[\mathcal{S}_n]\theta_n}M + \ldots \\ &= [\mathcal{S}_1]\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_n]\theta_n}M + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]\dot{\theta}_2 e^{[\mathcal{S}_2]\theta_2}\cdots e^{[\mathcal{S}_n]\theta_n}M + \ldots \end{aligned}$$

Also,

$$T^{-1} = M^{-1}e^{-[\mathcal{S}_n]\theta_n}\cdots e^{-[\mathcal{S}_1]\theta_1}.$$

Multiplying $\dot{T}$ and T^{-1}, we have

$$[\mathcal{V}_s] = [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \ldots.$$

The above can also be expressed in vector form by means of the adjoint mapping:

$$\mathcal{V}_s = \mathcal{S}_1\dot{\theta}_1 + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)\dot{\theta}_2 + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)\dot{\theta}_3 + \ldots \tag{6.2}$$

Observe that $\mathcal{V}_s$ is a sum of n spatial velocities of the form

$$\mathcal{V}_s = \mathcal{V}_{s1}(\theta)\dot{\theta}_1 + \ldots + \mathcal{V}_{sn}(\theta)\dot{\theta}_n, \tag{6.3}$$

where each $\mathcal{V}_{si}(\theta) = (\omega_{si}(\theta), v_{si}(\theta))$ depends explictly on the joint values $\theta \in \mathbb{R}^n$. In matrix form,

$$\begin{aligned} \mathcal{V}_s &= \left[\begin{array}{cccc} \mathcal{V}_{s1}(\theta) & \mathcal{V}_{s2}(\theta) & \cdots & \mathcal{V}_{sn}(\theta) \end{array}\right] \left[\begin{array}{c} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{array}\right] \\ &= J_s(\theta)\dot{\theta}. \end{aligned} \tag{6.4}$$

The matrix $J_s(\theta)$ is said to be the **Jacobian** in fixed (**space**) frame coordinates, or more simply the **space Jacobian**.

Definition 6.1. Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T = e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_n]\theta_n}M. \tag{6.5}$$

The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector spatial velocity $\mathcal{V}_s$ via $\mathcal{V}_s = J_s(\theta)\dot{\theta}$. The i-th column of $J_s(\theta)$ is

$$\mathcal{V}_{si}(\theta) = Ad_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i), \tag{6.6}$$

for $i = 2,\ldots,n$, with the first column $\mathcal{V}_{s1}(\theta) = \mathcal{S}_1$. $\square$

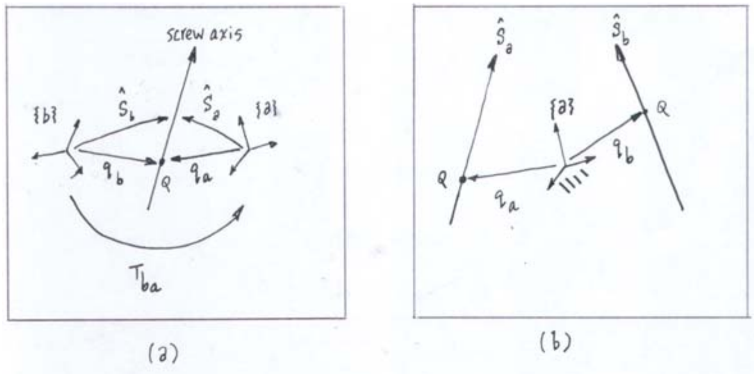

Figure 6.1: Physical interpretation of the screw adjoint transformation $Ad_{T_{ba}}$: (a) describing the same screw motion in terms of two different reference frames $\{a\}$ and $\{b\}$; (b) An initial screw axis displaced a transformation T_{ba}.

To understand the physical meaning behind the columns of $J_s(\theta)$, recall from Chapter 3 that if $\mathcal{S}_a = (\omega_a, v_a)$ is the vector describing a screw motion in frame $\{a\}$ coordinates, and $\mathcal{S}_b = (\omega_b, v_b)$ is a vector describing the same screw motion in frame $\{b\}$ coordinates, then $\mathcal{S}_b$ and $\mathcal{S}_a$ are related by $\mathcal{S}_b = \mathrm{Ad}_{T_{ba}}(\mathcal{S}_a)$ (see Figure 6.1-(a)). Another physical interpretation of this transformation is from the perspective of reference frame $\{a\}$ only. Referring to Figure 6.1-(b), suppose the vector $\mathcal{S}_a$ describes the initial screw axis with respect to frame $\{a\}$, and the vector $\mathcal{S}_b$ describes the screw axis after it has undergone a rigid body displacement T_{ba}. It follows that

$$\omega_b = R_{ba}\omega_a. \tag{6.7}$$

The point q on the screw axis is now displaced from its initial location q_a to

$$q_b = T_{ba}q_a = R_{ba}q_a + p_{ba}. \tag{6.8}$$

Then from the definition $v_b = -\omega_b \times q_b + h_b\omega_b$, where $h_b = h_a$ (the screw pitch is a scalar quantity, and hence independent of the choice of reference frames), we have

$$\begin{aligned} v_b &= -\omega_b \times q_b + h_b\omega_b \\ &= -R_{ba}\omega_a \times (R_{ba}q_a + p_{ba}) + h_b R_{ba}\hat{\omega}_a \\ &= R_{ba}\left(-[\omega_a]q_a + h_a\omega_a\right) - R_{ba}[\omega_a]R_{ba}^T p_{ba} \\ &= R_{ba}v_a + [p_{ba}]R_{ba}\omega_a, \end{aligned}$$

where in the last two lines we have again made use of the matrix identity $R[\omega]R^T = [R\omega]$ for $R \in SO(3)$ and $\omega \in \mathbb{R}^3$. Equations (6.7) and (6.9) can be combined in the form

$$\begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R_{ba} & 0 \\ [p_{ba}]\,R_{ba} & R_{ba} \end{bmatrix} \begin{bmatrix} \omega_a \\ v_a \end{bmatrix}, \tag{6.9}$$

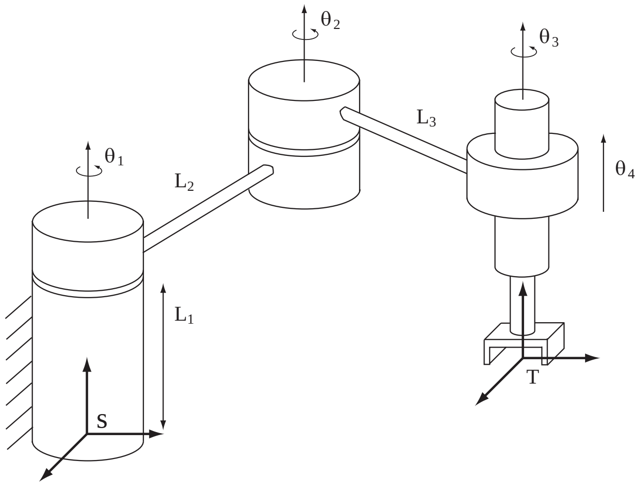

Figure 6.2: Space Jacobian for a spatial *RRRP* chain.

which is precisely $\mathcal{S}_b = \mathrm{Ad}_{T_{ba}}(\mathcal{S}_a)$.

Returning now to the equation for the space Jacobian (6.2), observe that the i-th term of the right-hand side of (6.2) is of the form $\mathrm{Ad}_{T_{i-1}}(\mathcal{S}_i)$, where $T_{i-1} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}$; recall here that $\mathcal{S}_i$ is the screw vector describing the i-th joint axis in terms of the fixed frame with the robot in its zero position. $\mathrm{Ad}_{T_{i-1}}(\mathcal{S}_i)$ can therefore be viewed as the screw vector describing the i-th joint axis after it undergoes the rigid body displacement T_{i-1}. But physically this is the same as moving the first $i-1$ joints from their zero position to the current values $\theta_1, \ldots, \theta_{i-1}$. Therefore, the i-th column $\mathcal{V}_{si}(\theta)$ of $J_s(\theta)$ is simply the screw vector describing joint axis i, expressed in fixed frame coordinates as a function of the joint variables $\theta_1, \ldots, \theta_{i-1}$.

In summary, the procedure for determining the columns of $J_s(\theta)$ is similar to that for deriving the $\mathcal{S}_i$ in the product of exponentials formula $e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M$: each column $\mathcal{V}_{si}(\theta)$ is the screw vector describing joint axis i, expressed in fixed frame coordinates, but for arbitrary θ rather than $\theta = 0$.

Example: Space Jacobian for a Spatial RRRP Chain

We now illustrate the procedure for finding the space Jacobian for the spatial *RRRP* chain of Figure 6.2. Denote the i-th column of $J_s(\theta)$ by $\mathcal{V}_i = (\omega_i, v_i)$.

- Observe that ω_1 is constant and in the $\hat{z}$-direction: $\omega_1 = (0, 0, 1)$. Picking q_1 to be the origin, $v_1 = (0, 0, 0)$.
- ω_2 is also constant in the $\hat{z}$-direction, so $\omega_2 = (0, 0, 1)$. Pick $q_2 = (L_1 c_1, L_1 s_1, 0)$, where $c_1 = \cos\theta_1$, $s_1 = \sin\theta_1$. Then $v_2 = -\omega_2 \times q_2 = (L_1 s_1, -L_1 c_1, 0)$.
- The direction of ω_3 is always fixed in the $\hat{z}$-direction regardless of the values of θ_1 and θ_2, so $\omega_3 = (0, 0, 1)$. Picking $q_3 = (L_1 c_1 + L_2 c_{12}, L_1 s_1 +$

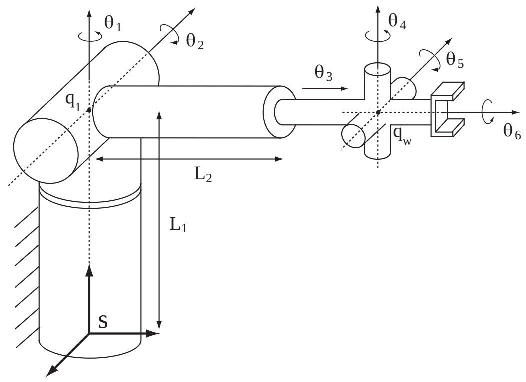

Figure 6.3: Space Jacobian for the spatial $RRPRRR$ chain.

$L_2 s_{12}, 0)$, where $c_{12} = \cos(\theta_1 + \theta_2)$, $s_{12} = \sin(\theta_1 + \theta_2)$, it follows that $v_3 = (L_1 s_1 + L_2 s_{12}, -L_1 c_1 - L_2 c_{12}, 0)$.

- Since the final joint is prismatic, $\omega_4 = (0,0,0)$, and the joint axis direction is given by $v_4 = (0,0,-1)$.

The space Jacobian is therefore

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & L_1 s_1 & L_1 s_1 + L_2 s_{12} & 0 \\ 0 & -L_1 c_1 & -L_1 c_1 - L_2 c_{12} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Example: Space Jacobian for Spatial RRPRRR Chain

We now derive the space Jacobian for the spatial $RRPRRR$ chain of Figure 6.3. The base frame is chosen as shown in the figure.

- The first joint axis is in the direction $\omega_1 = (0,0,1)$. Picking $q_1 = (0,0,L_1)$, we get $v_1 = -\omega_1 \times q_1 = (0,0,0)$.
- The second joint axis is in the direction $\omega_2 = (-c_1, -s_1, 0)$. Picking $q_2 = (0,0,L_1)$, we get $v_2 = -\omega_2 \times q_2 = (L_1 s_1, -L_1 c_1, 0)$.
- The third joint is prismatic, so $\omega_3 = (0,0,0)$. The direction of the prismatic joint axis is given by

$$v_3 = \mathrm{Rot}(\hat{z}, \theta_1)\mathrm{Rot}(\hat{x}, -\theta_2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1 c_2 \\ c_1 c_2 \\ -s_2 \end{bmatrix}.$$

- Now consider the wrist portion of the chain. The wrist center is located at the point

$$q_w = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + \mathrm{Rot}(\hat{z},\theta_1)\mathrm{Rot}(\hat{x},-\theta_2) \begin{bmatrix} 0 \\ L_1+\theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(L_2+\theta_3)s_1c_2 \\ (L_2+\theta_3)c_1c_2 \\ L_1-(L_2+\theta_3)s_2 \end{bmatrix}.$$

Observe that the directions of the wrist axes depend on θ_1, θ_2, and the preceding wrist axes. These are

$$\begin{aligned} \omega_4 &= \mathrm{Rot}(\hat{z},\theta_1)\mathrm{Rot}(\hat{x},-\theta_2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_1s_2 \\ c_1s_2 \\ c_2 \end{bmatrix} \\ \omega_5 &= \mathrm{Rot}(\hat{z},\theta_1)\mathrm{Rot}(\hat{x},-\theta_2)\mathrm{Rot}(\hat{z},\theta_4)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1c_4+s_1c_2s_4 \\ -s_1c_4-c_1c_2s_4 \\ s_2s_4 \end{bmatrix} \\ \omega_6 &= \mathrm{Rot}(\hat{z},\theta_1)\mathrm{Rot}(\hat{x},-\theta_2)\mathrm{Rot}(\hat{z},\theta_4)\mathrm{Rot}(\hat{x},-\theta_5)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -c_5(s_1c_2c_4+c_1s_4)+s_1s_2s_5 \\ c_5(c_1c_2c_4-s_1s_4)-c_1s_2s_5 \\ -s_2c_4c_5-c_2s_5 \end{bmatrix}. \end{aligned}$$

The space Jacobian can now be computed and written in matrix form as follows:

$$J_s(\theta) = \begin{bmatrix} \omega_1 & \omega_2 & 0 & \omega_4 & \omega_5 & \omega_6 \\ 0 & -\omega_2\times q_2 & v_3 & -\omega_4\times q_w & -\omega_5\times q_w & -\omega_6\times q_w \end{bmatrix}.$$

Although the resulting Jacobian is quite complicated, note that we were able to calculate the entire Jacobian without having to explicitly differentiate the forward kinematic map.

6.1.2 Body Jacobian

In the previous section we derived the relationship between the joint rates and $[\mathcal{V}_s] = \dot{T}T^{-1}$, the end-effector's spatial velocity expressed in fixed frame coordinates. Here we derive the relationship between the joint rates and $[\mathcal{V}_b] = T^{-1}\dot{T}$, the end-effector spatial velocity in end-effector frame coordinates. For this purpose it will be more convenient to express the forward kinematics in the alternate product of exponentials form:

$$T(q) = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}. \quad (6.10)$$

Computing $\dot{T}$,

$$\begin{aligned} \dot{T} &= Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}(\frac{d}{dt}e^{[\mathcal{B}_n]\theta_n}) + Me^{[\mathcal{B}_1]\theta_1}\cdots(\frac{d}{dt}e^{[\mathcal{B}_{n-1}]\theta_{n-1}})e^{[\mathcal{B}_n]\theta_n} + \ldots \\ &= Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_n]\theta_n}[\mathcal{B}_n]\dot{\theta}_n + Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + \ldots \\ &\quad + Me^{[\mathcal{B}_1]\theta_1}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1. \end{aligned}$$

Also,

$$T^{-1} = e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_1]\theta_1} M^{-1}.$$

Multiplying T^{-1} and $\dot{T}$,

$$\begin{aligned}[\mathcal{V}_b] &= [\mathcal{B}_n]\dot{\theta}_n + e^{-[\mathcal{B}_n]\theta_n}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + \ldots \\ &\quad + e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_2]\theta_2}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1,\end{aligned}$$

or in vector form,

$$\mathcal{V}_b = \mathcal{B}_n\dot{\theta}_n + \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1})\dot{\theta}_{n-1} + \ldots + \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_2]\theta_2}}(\mathcal{B}_1)\dot{\theta}_1. \quad (6.11)$$

$\mathcal{V}_b$ can therefore be expressed as a sum of n spatial velocities, i.e.,

$$\mathcal{V}_b = \mathcal{V}_{b1}(\theta)\dot{\theta}_1 + \ldots + \mathcal{V}_{bn}(\theta)\dot{\theta}_n, \quad (6.12)$$

where each $\mathcal{V}_{bi}(\theta) = (\omega_{bi}(\theta), v_{bi}(\theta))$ depends explictly on the joint values θ. In matrix form,

$$\begin{aligned}\mathcal{V}_b &= \begin{bmatrix} \mathcal{V}_{b1}(\theta) & \mathcal{V}_{b2}(\theta) & \cdots & \mathcal{V}_{bn}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \\ &= J_b(\theta)\dot{\theta}.\end{aligned} \quad (6.13)$$

The matrix $J_b(q)$ is the Jacobian in the end-effector (or **body**) frame coordinates, or more simply the **body Jacobian**.

Definition 6.2. Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}. \quad (6.14)$$

The **body Jacobian** $J_b(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector spatial velocity $\mathcal{V}_b = (\omega_b, v_b)$ via

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}. \quad (6.15)$$

The i-th column of $J_b(\theta)$ is given by

$$\mathcal{V}_{b,i}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i), \quad (6.16)$$

for $i = n, n-1, \ldots, 2$, with $\mathcal{V}_{bn}(\theta) = \mathcal{B}_n$. □

Analogous to the columns of the space Jacobian, a similar physical interpretation can also be given to the columns of $J_b(\theta)$: each column $\mathcal{V}_{bi}(\theta) = (\omega_{bi}(\theta), v_{bi}(\theta))$ of $J_b(\theta)$ is the screw vector for joint axis i, expressed in coordinates of the end-effector frame rather than the fixed frame. The procedure for determining the columns of $J_b(\theta)$ is similar to the procedure for deriving the forward kinematics in the product of exponentials form $Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}$, the only difference being that each of the joint screws are derived for arbitrary θ rather than $\theta = 0$.

6.1.3 Relationship between the Space and Body Jacobian

If we denote the fixed frame by $\{s\}$, and the robot arm's end-effector frame by $\{b\}$, then the forward kinematics can be written $T_{sb}(\theta)$. The spatial velocity of the tip frame can be written in terms of the fixed and end-effector frame coordinates as

$$[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1} \quad (6.17)$$

$$[\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb}, \quad (6.18)$$

with $\mathcal{V}_s$ and $\mathcal{V}_b$ related by $\mathcal{V}_s = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$. $\mathcal{V}_s$ and $\mathcal{V}_b$ are also related to their respective Jacobians via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} \quad (6.19)$$

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}. \quad (6.20)$$

Equation (6.19) can therefore be written

$$\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b) = J_s(\theta)\dot{\theta}. \quad (6.21)$$

Applying $\mathrm{Ad}_{T_{bs}}$ to both sides of (6.21), and using the general property $\mathrm{Ad}_M \cdot \mathrm{Ad}_N = \mathrm{Ad}_{MN}$ of the adjoint map, we obtain

$$\begin{aligned} \mathrm{Ad}_{T_{bs}}(\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)) &= \mathrm{Ad}_{T_{bs}T_{sb}}(\mathcal{V}_b) \\ &= \mathcal{V}_b \\ &= \mathrm{Ad}_{T_{bs}}\left(J_s(q)\dot{\theta}\right). \end{aligned}$$

Since we also have $\mathcal{V}_b = J_b(\theta)\dot{\theta}$ for all $\dot{\theta}$, it follows that $J_s(\theta)$ and $J_b(\theta)$ are related by

$$J_s(\theta) = \mathrm{Ad}_{T_{sb}}\left(J_b(\theta)\right). \quad (6.22)$$

Writing $\mathrm{Ad}_{T_{sb}}$ in 6×6 matrix form $[\mathrm{Ad}_{T_{sb}}]$, the above can also be expressed as

$$J_s(\theta) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta). \quad (6.23)$$

The body Jacobian can in turn be obtained from the space Jacobian via

$$J_b(\theta) = Ad_{T_{bs}}\left(J_s(\theta)\right) = [\mathrm{Ad}_{T_{bs}}]J_s(\theta). \quad (6.24)$$

6.2 Statics of Open Chains

A rigid body is said to be in **static equilibrium** if it is motionless, and the resultant forces and moments applied to the body are all zero. Let us briefly review the notion of moments, by considering a force $\mathbf{f}$ acting on a rigid body. If the rigid body's center of mass does not lie on the line of action of the force, then the force will cause the rigid body to rotate; this rotation is caused by a moment. More precisely, the moment *mbox*$\mathbf{m}$ generated by $\mathbf{f}$ about some reference point $\mathbf{P}$ in physical space is defined to be the cross product

$$\vec{m} = \mathbf{r} \times \mathbf{f}, \tag{6.25}$$

where $\mathbf{r}$ is the vector from $\mathbf{P}$ to the point on the rigid body at which the force is applied. For a rigid body subject to a collection of forces and moments, if both the sum of the forces and sum of the moments are zero, then the body will be stationary, and is said to be in static equilibrium. When summing moments it is important that each of the moments be expressed with respect to the same reference point $\mathbf{P}$.

A robot arm is said to be in static equilibrium if all of its links are in static equilibrium. In this section we shall examine, for robot arms that are in static equilibrium, the relationship between any external forces and moments applied at the end-effector, and the forces and torques experienced at each of the joints. Such a situation arises, for example, when a six degree-of-freedom arm is pushing against an immobile wall.

We first review spatial forces (also referred to as wrenches in the classical screw theory literature), which are obtained by merging forces and moments into a single six-dimensional quantity, much like spatial velocities are obtained by merging linear and angular velocities into a single six-dimensional vector. As we did for spatial velocities, we shall also examine how spatial forces transform under a change of reference frames.

We then review the principle of virtual work, but expressed in terms of spatial velocities and spatial forces. Applying the virtual work principle to a robot arm assumed to be in static equilibrium leads to our main result, which states that any external spatial forces applied at the end-effector frame are linearly related to the torques experienced at the joints.

6.2.1 Spatial Forces

Just as we found it advantageous to merge a moving frame's angular velocity $\omega \in \mathbb{R}^3$ with its linear velocity $v \in \mathbb{R}^3$ into a single six-dimensional spatial velocity $\mathcal{V} = (\omega, v)$, for the same reasons it will be useful to analogously define a six-dimensional **spatial force**, by merging a three-dimensional force vector $f \in \mathbb{R}^3$ with a three-dimensional moment vector $m \in \mathbb{R}^3$ as follows:

$$\mathcal{F} = \begin{bmatrix} m \\ f \end{bmatrix}, \tag{6.26}$$

which for notational convenience we will also write $\mathcal{F} = (m, f)$.

Let us find explicit expressions for the spatial force in terms of specific reference frames. For this purpose consider a rigid body with a moving (body) frame $\{b\}$ attached. Expressing everything in terms of the $\{b\}$ frame, let $f_b \in \mathbb{R}^3$ denote a force vector that is applied to a point p on the body. This force then generates a moment with respect to the $\{b\}$ frame origin; in $\{b\}$ frame coordinates, this moment is

$$m_b = r_b \times f_b, \tag{6.27}$$

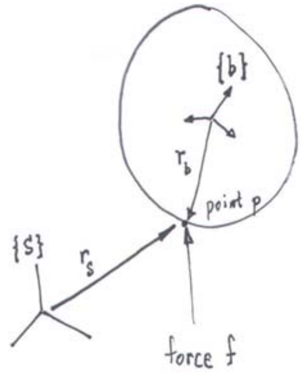

Figure 6.4: Relation between $\mathcal{F}_b$ and $\mathcal{F}_s$.

where $r_b \in \mathbb{R}^3$ is the vector from the $\{b\}$ frame origin to p. We shall pair the force f_b and moment m_b into a single six-dimensional spatial force $\mathcal{F}_b = (m_b, f_b)$, and refer to it as the **spatial force in body frame coordinates**.

Suppose we now wish to express the force and moment in terms of the fixed (space) frame $\{s\}$. Let $f_s \in \mathbb{R}^3$ denote the force vector being applied to point p of the rigid body, this time expressed in $\{s\}$ frame coordinates. The moment generated by this force with respect to the $\{s\}$ frame origin is, again in $\{s\}$ frame coordinates,

$$m_s = r_s \times f_s, \tag{6.28}$$

where $r_s \in \mathbb{R}^3$ is the vector from the $\{s\}$ frame origin to p. As we did for $\mathcal{F}_b$, let us also bundle f_s and m_s into the six-dimensional spatial force $\mathcal{F}_s = (m_s, f_s)$, and refer to it as the **spatial force in space frame coordinates**.

We now determine the relation between $\mathcal{F}_b = (m_b, f_b)$ and $\mathcal{F}_s = (m_s, f_s)$. Referring to Figure 6.4, denote the transformation T_{sb} by

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix}.$$

Pretty clearly $f_b = R_{bs} f_s$, which with the benefit of hindsight we shall write in the somewhat unconventional form

$$f_b = R_{sb}^T f_s. \tag{6.29}$$

The moment m_b is given by $r_b \times f_b$, where $r_b = R_{bs}(r_s - p_{sb})$; this follows from the fact that the $r_s - p_{sb}$ is expressed in $\{s\}$ frame coordinates, and must be transformed to $\{b\}$ frame coordinates via multiplication by R_{bs}. Again with hindsight, we shall write

$$r_b = R_{sb}^T (r_s - p_{sb}).$$

The moment $m_b = r_b \times f_b$ can now be written in terms of f_s and m_s as

$$\begin{aligned} m_b &= R_{sb}^T(r_s - p_{sb}) \times R_{sb}^T f_s \\ &= [R_{sb}^T r_s] R_{sb}^T f_s - [R_{sb}^T p_{sb}] R_{sb}^T f_s \\ &= R_{sb}^T [r_s] f_s - R_{sb}^T [p_{sb}] f_s \\ &= R_{sb}^T m_s + R_{sb}^T [p_{sb}]^T f_s, \end{aligned} \tag{6.30}$$

where in the last line we make use of the fact that $[p_{sb}]^T = -[p_{sb}]$. Writing both m_b and f_b in terms of m_s and f_s, we have, from Equations (6.29) and (6.30),

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} R_{sb} & 0 \\ [p_{sb}]\, R_{sb} & R_{sb} \end{bmatrix}^T \begin{bmatrix} m_s \\ f_s \end{bmatrix}, \tag{6.31}$$

or in terms of spatial forces and the adjoint map,

$$\mathcal{F}_b = \mathrm{Ad}_{T_{sb}}^T(\mathcal{F}_s) = [\mathrm{Ad}_{T_{sb}}]^T \mathcal{F}_s. \tag{6.32}$$

We see that under a change of reference frames, spatial velocities transform under the adjoint map, whereas spatial forces transform under the adjoint transpose map. In fact, the introduction of the rigid body is superfluous; the above relation holds for all spatial forces described in terms of two different reference frames. The following proposition formally states this result.

Proposition 6.1. *Given a force* $\mathbf{f}$*, let* $\mathbf{m}$ *be the moment generated by* $\mathbf{f}$ *with respect to some point* $\mathbf{P}$ *in physical space. Given a reference frame* $\{a\}$*, let* $f_a \in \mathbb{R}^3$ *and* $m_a = r_a \times f_a \in \mathbb{R}^3$ *be representations of* $\mathbf{f}$ *and* $\mathbf{m}$ *in frame* $\{a\}$ *coordinates, where* $r_a \in \mathbb{R}^3$ *is the vector from the* $\{a\}$ *frame origin to* p*, also expressed in* $\{a\}$ *frame coordinates. Similarly, given another reference frame* $\{b\}$*, let* $f_b \in \mathbb{R}^3$ *and* $m_b = r_b \times f_b \in \mathbb{R}^3$ *be representations of* $\mathbf{f}$ *and* $\mathbf{m}$ *frame* $\{b\}$ *coordinates, where* $r_b \in \mathbb{R}^3$ *is the vector from the* $\{b\}$ *frame origin to* p*, also expressed in* $\{b\}$ *frame coordinates. Defining the spatial forces* $\mathcal{F}_a = (r_a \times f_a, f_a)$ *and* $\mathcal{F}_b = (r_b \times f_b, f_b)$*,* $\mathcal{F}_a$ *and* $\mathcal{F}_b$ *are related by*

$$\mathcal{F}_b = Ad_{T_{ab}}^T(\mathcal{F}_a) = [Ad_{T_{ab}}]^T \mathcal{F}_a \tag{6.33}$$

$$\mathcal{F}_a = Ad_{T_{ba}}^T(\mathcal{F}_b) = [Ad_{T_{ba}}]^T \mathcal{F}_b. \tag{6.34}$$

6.2.2 Static Analysis and the Virtual Work Principle

When a robot arm is in static equilibrium, it turns out that the Jacobian of the forward kinematics also relates any external forces and torques applied at the end-effector to the torques experienced at each of the joints. This can be shown by appealing to the **Principle of Virtual Work**, which we now describe. Fix a reference frame, and consider a rigid body moving with velocity v and angular velocity ω, and subject to a resultant force f and resultant moment m. The work done by the rigid body over some time interval $[t_0, t_1]$ is given by the integral

$$\text{Work} = \int_{t_0}^{t_1} f^T v + m^T \omega \, dt \tag{6.35}$$

In terms of spatial forces and velocities, (6.35) can also be expressed as

$$\text{Work} = \int_{t_0}^{t_1} \mathcal{F}^T \mathcal{V} \, dt, \tag{6.36}$$

where $\mathcal{F} = (m, f)$ and $\mathcal{V} = (\omega, v)$. The work of a system of rigid bodies is simply the sum of the work done by each of the rigid bodies.

For a single rigid body, suppose that the resultant force and moment are applied to the body over an infinitesimal time interval δt, resulting in an infinitesimally small displacement of the body. If the body is in static equilibrium, the body is stationary and thus will produce no work. This infinitesimal displacement over δt can still be thought of as a virtual displacement. The virtual work principle states that under static equilibrium, the work of any external forces and moments acting on a rigid body is always zero for any admissible virtual displacement of the body. This principle also extends to robot arms, and more generally to any system of connected rigid bodies: for any admissible virtual displacement of the system (i.e., one that does not violate any kinematic constraints), the total virtual work of the external forces and moments acting on the system is zero.

Now consider an n-link robot arm assumed to be in static equilibrium, and suppose a force and moment are applied to the tip. For now all quantities are defined in terms of the end-effector (body) frame: let $\mathcal{F}_b = (m_b, f_b)$ be an external spatial force applied over some infinitesimal time interval $[t_0, t_0 + \delta t]$, and $\mathcal{V}_b = (\omega_b, v_b)$ be the (instantaneous) spatial velocity of the end-effector. The net virtual work done by the robot is given by Equation (6.35). Assuming the robot is lossless, from the virtual work principle this infinitesimal work should be the same as that produced by any torques applied at the joints:

$$\text{Virtual Work} = \int_{t_0}^{t_0+\delta t} \mathcal{F}_b^T \mathcal{V}_b \, dt = \int_{t_0}^{t_0+\delta t} \tau^T \dot{\theta} \, dt,$$

where $\dot{\theta} \in \mathbb{R}^n$ is the vector of joint velocities, and $\tau \in \mathbb{R}^n$ is the vector of joint torques. Since $\mathcal{V}_b = J_b(\theta)\dot{\theta}$, we have

$$\int_{t_0}^{t_0+\delta t} F_b^T J_b(\theta)\dot{\theta} \, dt = \int_{t_0}^{t_0+\delta t} \tau^T \dot{\theta} \, dt.$$

Since this equality must hold over all intervals $[t_0, t_0 + \delta t]$, the integrands must be equal:

$$\mathcal{F}_b^T J_b(\theta)\dot{\theta} = \tau^T \dot{\theta}.$$

Moreover, since the above equality must hold for all admissible virtual displacements $\dot{\theta} \, \delta t$—in this case $\dot{\theta}$ can be arbitrary—it follows that

$$\tau = J_b^T(\theta)\mathcal{F}_b. \tag{6.37}$$

Let us replicate the derivation of $\tau = J_b^T(\theta)\mathcal{F}_b$, but this time expressing all quantities in terms of the fixed (space) frame. Let $\mathcal{V}_s = (\omega_s, v_s)$ denote the spatial velocity of the end-effector, and $\mathcal{F}_s = (m_s, f_s)$ the spatial force applied at the end-effector frame origin, all expressed in fixed frame coordinates. Here f_s is the external applied force expressed in fixed frame coordinates, while m_s is the external applied moment about the fixed frame origin. Recalling from the earlier section on spatial forces that $\mathcal{F}_b$ and $\mathcal{F}_s$ are related by $\mathcal{F}_b = [\mathrm{Ad}_{T_sb}]^T \mathcal{F}_s$, and that $J_b(\theta)$ and $J_s(\theta)$ are further related by $J_b(\theta) = [\mathrm{Ad}_{T_{bs}}]J_s(\theta)$, Equation (6.37) can be rewritten

$$\begin{aligned} \tau &= J_b^T(\theta)\mathcal{F}_b = \left([\mathrm{Ad}_{T_{bs}}]J_s(\theta)\right)^T [\mathrm{Ad}_{T_sb}]^T \mathcal{F}_s \\ &= J_s^T(\theta)\left([\mathrm{Ad}_{T_sb}][Ad_{T_{bs}}]\right)^T \mathcal{F}_s \\ &= J_s^T(\theta)\mathcal{F}_s. \end{aligned} \tag{6.38}$$

We can therefore write the statics relation in the general form

$$\tau = J^T(\theta)\mathcal{F}, \tag{6.39}$$

with the understanding that $J(\theta)$ and $\mathcal{F}$ are expressed in terms of the same frame. Often in robotics one is interested in determining what joint torques are necessary, under static equilibrium assumptions, to produce a given desired $\mathcal{F}$; the static relation provides an explicit answer to this question.

One could also ask the opposite question, namely, what is the spatial force at the tip generated by a given joint torque? If J^T is a square invertible matrix, then clearly $\mathcal{F} = J^{-T}(\theta)\tau$. However, if the dimension of the joint vector n is greater than the dimension of $\mathcal{F}$ (six), then the inverse of J^T does not exist. What this implies physically is that the robot arm has extra degrees of freedom. Because of the extra degrees of freedom, some of the robot's links can move even when the end-effector is fixed (for example, the planar four-bar linkage can be regarded as a $3R$ planar open chain with its tip fixed to the base joint). Internal motions are generated as a result of the applied joint torques, and the static equilibrium condition is no longer satisfied. Robots whose joint degrees of freedom exceed the dimension of its task space are called **kinematically redundant**; in the next chapter we shall examine the inverse kinematics of kinematically redundant robot arms.

6.3 Singularities

The forward kinematics Jacobian also allows us to identify postures at which the robot's end-effector loses the ability to move instantaneously in a certain direction, or rotate instantaneously about certain axes; such a posture is called a **kinematic singularity**, or simply a **singularity**. Mathematically a singular posture of a robot arm is one in which the Jacobian loses rank. To understand why, consider the body Jacobian $J_b(\theta)$, whose columns are denoted $\mathcal{V}_{bi}$, $i =$

$1, \ldots, n$. Then

$$\begin{aligned} \mathcal{V}_b &= \left[\begin{array}{cccc} \mathcal{V}_{b1}(\theta) & \mathcal{V}_{b2}(\theta) & \cdots & \mathcal{V}_{bn}(\theta) \end{array} \right] \left[\begin{array}{c} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{array} \right] \\ &= \mathcal{V}_{b1}(\theta)\dot{\theta}_1 + \ldots + \mathcal{V}_{bn}(\theta)\dot{\theta}_n. \end{aligned}$$

Thus, the set of all possible instantaneous spatial velocities of the tip frame is given by a linear combination of the $\mathcal{V}_{bi}$. As long as $n \geq 6$, the maximum rank that $J_b(\theta)$ can attain is six. Singular postures correspond to those values of θ at which the rank of $J_b(\theta)$ drops below six; at such postures the tip frame loses to the ability to generate instantaneous spatial velocities in in one or more dimensions.

The mathematical definition of a kinematic singularity is independent of the choice of body or space Jacobian. To see why, recall the relationship between $J_s(\theta)$ and $J_b(\theta)$: $J_s(q) = [Ad_{T_{sb}}(J_b(\theta)) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta)$, or more explicitly,

$$J_s(\theta) = \left[\begin{array}{cc} R_{sb} & 0 \\ [p_{sb}]\, R_{sb} & R_{sb} \end{array} \right] J_b(\theta).$$

Then the rank of $J_s(\theta)$ is equal to the rank of $[\mathrm{Ad}_{T_{sb}}]J_b(\theta)$. We now claim that the matrix $[\mathrm{Ad}_{T_{sb}}]$ is always invertible. This can be established by examining the linear equation

$$\left[\begin{array}{cc} R_{sb} & 0 \\ [p_{sb}]\, R_{sb} & R_{sb} \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] = 0.$$

Its unique solution is $x = y = 0$, implying that the matrix $[\mathrm{Ad}_{T_{sb}}]$ is invertible. Since multiplying any matrix by an invertible matrix does not change its rank, it follows that

$$\text{rank } J_s(\theta) = \text{rank } J_b(\theta),$$

as claimed; singularities of the space and body Jacobian are the one and the same.

Kinematic singularities are also independent of the choice of fixed frame. In some sense this is rather obvious—choosing a different fixed frame is equivalent to simply relocating the robot arm, which should have absolutely no effect on whether a particular posture is singular or not. This obvious fact can be verified by referring to Figure 6.5-(a). The forward kinematics with respect to the original fixed frame is denoted $T(\theta)$, while the forward kinematics with respect to the relocated fixed frame is denoted $T'(\theta) = PT(\theta)$, where $P \in SE(3)$ is constant. Then the body Jacobian of $T'(\theta)$, denoted $J_b'(\theta)$, is obtained from $T'^{-1}\dot{T}'$. A simple calculation reveals that

$$T'^{-1}\dot{T}' = (T^{-1}P^{-1})(P\dot{T}) = T^{-1}\dot{T},$$

i.e., $J_b'(\theta) = J_b(\theta)$, so that the singularities of the original and relocated robot arms are the same.

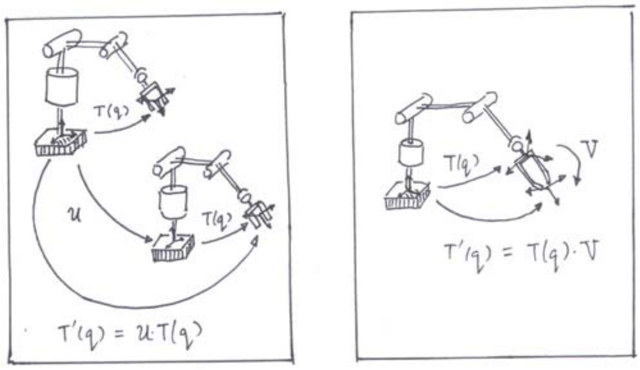

Figure 6.5: Kinematic singularities are invariant with respect to choice of fixed and end-effector frames. (a) Choosing a different fixed frame, which is equivalent to relocating the base of the robot arm; (b) Choosing a different end-effector frame.

Somewhat less obvious is the fact that kinematic singularities are also independent of the choice of end-effector frame. Referring to Figure 6.5-(b), suppose the forward kinematics for the original end-effector frame is given by $T(\theta)$, while the forward kinematics for the relocated end-effector frame is $T'(\theta) = T(\theta)Q$, where $Q \in SE(3)$ is constant. This time looking at the space Jacobian—recall that singularities of $J_b(\theta)$ coincide with those of $J_s(\theta)$—let $J_s'(\theta)$ denote the space Jacobian of $T'(\theta)$. A simple calculation reveals that

$$\dot{T}'T'^{-1} = (\dot{T}Q)(Q^{-1}T^{-1}) = \dot{T}T^{-1},$$

i.e., $J_s'(\theta) = J_s(\theta)$, so that kinematic singularities are invariant with respect to choice of end-effector frame.

In the remainder of this section we shall consider some common kinematic singularities that occur in six degree of freedom open chains with revolute and prismatic joints. We now know that either the space or body Jacobian can be used for our analysis; we shall use the space Jacobian in the examples below.

Case I: Two Collinear Revolute Joint Axes

The first case we consider is one in which two revolute joint axes are collinear (see Figure 6.6). Without loss of generality these joint axes can be labelled 1 and 2. The corresponding columns of the Jacobian are

$$\mathcal{V}_{s1}(\theta) = \begin{bmatrix} \omega_1 \\ -\omega_1 \times q_1 \end{bmatrix}, \quad \mathcal{V}_{s2}(\theta) = \begin{bmatrix} \omega_2 \\ -\omega_2 \times q_2 \end{bmatrix}$$

Since the two joint axes are collinear, we must have $\omega_1 = \pm\omega_2$; let us assume the positive sign. Also, $\omega_i \times (q_1 - q_2) = 0$ for $i = 1, 2$. Then $\mathcal{V}_{s1} - \mathcal{V}_{s2} = 0$, which implies that $\mathcal{V}_{s1}$ and $\mathcal{V}_{s2}$ lie on the same line in six-dimensional space.

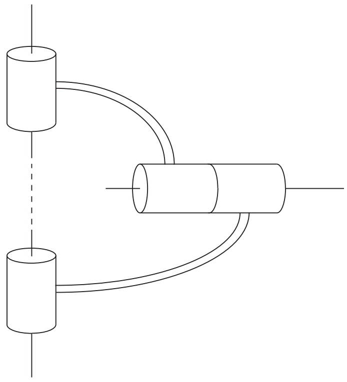

Figure 6.6: A kinematic singularity in which two joint axes are collinear.

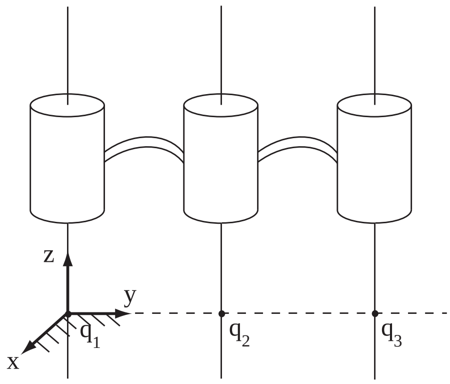

Figure 6.7: A kinematic singularity in which three revolute joint axes are parallel and coplanar.

Therefore, the set $\{\mathcal{V}_{s1}, \mathcal{V}_{s2}, \ldots, \mathcal{V}_{s6}\}$ cannot be linearly independent, and the rank of $J_s(\theta)$ must be less than six.

Case II: Three Coplanar and Parallel Revolute Joint Axes

The second case we consider is one in which three revolute joint axes are parallel, and also lie on the same plane (three coplanar axes—see Figure 6.7). Without loss of generality we label these as joint axes 1, 2, and 3. In this case we choose the fixed frame as shown in the figure; then

$$J_s(\theta) = \begin{bmatrix} \omega_1 & \omega_1 & \omega_1 & \cdots \\ 0 & -\omega_1 \times q_2 & -\omega_1 \times q_3 & \cdots \end{bmatrix}$$

and since q_2 and q_3 are points on the same unit axis, it is not difficult to verify that the above three vectors cannot be linearly independent.

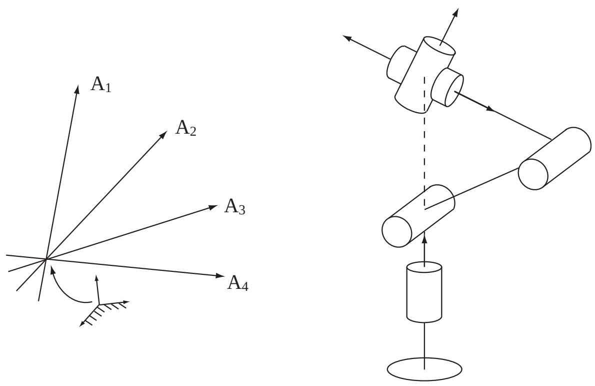

Figure 6.8: A kinematic singularity in which four revolute joint axes intersect at a common point.

Case III: Four Revolute Joint Axes Intersecting at a Common Point

Here we consider the case where four revolute joint axes intersect at a common point (Figure 6.8). Again, without loss of generality label these axes from 1 to 4. In this case we choose the fixed frame origin to be the common point of intersection, so that $q_1 = \ldots = q_4 = 0$. In this case

$$J_s(\theta) = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & \omega_4 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}.$$

The first four columns clearly cannot be linearly independent; one can be written as a linear combination of the other three. Such a singularity occurs, for example, when the wrist center of an elbow-type robot arm is directly above the shoulder.

Case IV: Four Coplanar Revolute Joints

Here we consider the case in which four revolute joint axes are coplanar. Again, without loss of generality label these axes from 1 to 4. Choose a fixed frame such that the joint axes all lie on the x-y plane; in this case the unit vector $\omega_i \in \mathbb{R}^3$ in the direction of joint axis i is of the form

$$\omega_i = \begin{bmatrix} \omega_{ix} \\ \omega_{iy} \\ 0 \end{bmatrix}.$$

Similarly, any reference point $q_i \in \mathbb{R}^3$ lying on joint axis i is of the form

$$q_i = \begin{bmatrix} q_{ix} \\ q_{iy} \\ 0 \end{bmatrix},$$

and subsequently

$$v_i = -\omega_i \times q_i = \begin{bmatrix} 0 \\ 0 \\ \omega_{iy} q_{ix} - \omega_{ix} q_{iy} \end{bmatrix}.$$

The first four columns of the space Jacobian $J_s(\theta)$ are

$$\begin{bmatrix} \omega_{1x} & \omega_{2x} & \omega_{3x} & \omega_{4x} \\ \omega_{1y} & \omega_{2y} & \omega_{3y} & \omega_{4y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_{1y}q_{1x} - \omega_{1x}q_{1y} & \omega_{2y}q_{2x} - \omega_{2x}q_{2y} & \omega_{3y}q_{3x} - \omega_{3x}q_{3y} & \omega_{4y}q_{4x} - \omega_{4x}q_{4y} \end{bmatrix}.$$

which clearly cannot be linearly independent.

Case V: Six Revolute Joints Intersecting a Common Line

The final case we consider is six revolute joint axes intersecting a common line. Choose a fixed frame such that the common line lies along the $\hat{z}$-axis, and select the intersection between this common line and joint axis i as the reference point $q_i \in \mathbb{R}^3$ for axis i; each q_i is thus of the form $q_i = (0, 0, q_{iz})$, and

$$v_i = -\omega_i \times q_i = (\omega_{iy} q_{iz}, -\omega_{ix} q_{iz}, 0)$$

$i = 1, \ldots, 6$. The space Jacobian $J_s(\theta)$ thus becomes

$$J_s(\theta) = \begin{bmatrix} \omega_{1x} & \omega_{2x} & \omega_{3x} & \omega_{4x} & \omega_{5x} & \omega_{6x} \\ \omega_{1y} & \omega_{2y} & \omega_{3y} & \omega_{4y} & \omega_{5y} & \omega_{6y} \\ \omega_{1z} & \omega_{2z} & \omega_{3z} & \omega_{4z} & \omega_{5z} & \omega_{6z} \\ \omega_{1y}q_{1z} & \omega_{2y}q_{2z} & \omega_{3y}q_{3z} & \omega_{4y}q_{4z} & \omega_{5y}q_{5z} & \omega_{6y}q_{6z} \\ -\omega_{1x}q_{1z} & -\omega_{2x}q_{2z} & -\omega_{3x}q_{3z} & -\omega_{4x}q_{4z} & -\omega_{5x}q_{5z} & -\omega_{6x}q_{6z} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is clearly singular.

6.4 Manipulability

In the previous section we saw that at a kinematic singularity, a robot's end-effector loses the ability to move or rotate in one or more directions. A kinematic singularity is a binary proposition—a particular configuration is either kinematically singular, or it isn't—and it is reasonable to ask whether it is possible to quantify the proximity of a particular configuration to a singularity. The answer is yes; in fact, one can even do better and quantify not only the proximity to a singularity, but also determine the directions in which the end-effector's ability to move is diminished, and to what extent. The **manipulabiity ellipsoid** allows one to geometrically visualize the directions in which the end-effector can

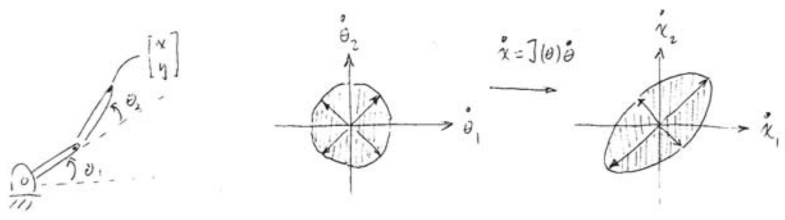

Figure 6.9: The manipulability ellipsoid for a planar $2R$ open chain.

move with the least "effort" (in a sense to be made precise below); directions that are orthogonal to these directions in contrast require the greatest effort.

We illustrate the notion of manipulability ellipsoids through the planar $2R$ open chain example of Figure 6.9. Considering only the Cartesian position of the tip, the velocity forward kinematics is of the form $\dot{x} = J(\theta), \dot{\theta}$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 & L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin\theta_1 & L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}. \tag{6.40}$$

Suppose the configuration θ is nonsingular, so that $J(\theta)$ is invertible. Since $J(\theta)$ is a linear transformation that maps joint velocities to tip velocities, one can conjecture that a unit circle in the space of joint velocities maps to an ellipsoid in the space of tip velocities. To see why, the unit circle is parametrized by the constraint $\|\dot{\theta}\|^2 = 1$; the same constraint expressed in terms of tip velocities is $\|J^{-1}(\theta)\dot{x}\|^2 = 1$. Denoting the elements of $J^{-1}(\theta)$ by

$$J^{-1}(\theta) = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the constraint on tip velocities becomes

$$\alpha\dot{x}_1^2 + \beta\dot{x}_1\dot{x}_2 + \gamma\dot{x}_2^2 = 1,$$

where $\alpha = a^2 + b^2$, $\beta = 2(ab + cd)$, $\gamma = b^2 + d^2$. As is well known this is the equation for an ellipse centered at the origin (provided $\beta^2 - 4\alpha\gamma < 0$, which it is).

The major axes of the ellipse indicate the directions in which the tip can move (or more technically, generate velocities) with the least amount of effort (effort here corresponding to input velocities). By the same reasoning, the minor axes indicate the directions of motion for which the greatest amount of effort is required. As the arm configuration approaches a kinematic singularity, the ellipsoid eventually collapses to a line directed along the direction of allowable motion. The proximity of a particular configuration to a singularity can be measured in several ways, e.g., by the ratio of the lengths between the major and minor axes, with a minimum value of 1 indicating that the tip can move uniformly easily in all directions—such configurations are also sometimes referred to as **isotropic** configurations. In the absence of any preferred directions of the end-effector, such an isotropic configuration would be a reasonable choice for a robot performing generic tasks.

The above formulation can be generalized to spatial open chains more or less straightforwardly; without working out the details of the derivation (which require some further results from linear algebra and finite dimensional optimization—some of these details are examined in the exercises), we now formulate the manipulability ellipsoid for an n degree of freedom open chain ($n \geq 6$). Let $J_b(\theta) \in \mathbb{R}^{6\times n}$ be the body Jacobian (the choice of body Jacobian is arbitrary—the space Jacobian could just as easily have been chosen), which can be partitioned into its angular and linear velocity components $J_\omega(\theta) \in \mathbb{R}^{3\times n}$ and $J_v(\theta) \in \mathbb{R}^{3\times n}$ as follows:

$$\omega_b = J_\omega(\theta)\dot{\theta} \qquad (6.41)$$

$$v_b = J_v(\theta)\dot{\theta}. \qquad (6.42)$$

At this point one may ask why we choose to partition the Jacobian in this way. The reason is that the notion of a manipulability ellipsoid in the six-dimensioal space of spatial velocities (ω, v) makes little if any sense—the physical units for angular velocities are different from those for linear velocities. Any ellipsoid that merges these physically different quantities will depend, ultimately, on the choice of length scale for physical space, which as is well known is arbitrary. On the other hand, ellipsoids restricted to the space of Cartesian velocities $v_b \in \mathbb{R}^3$ are quite meaningful (as are its counterpart ellipsoids restricted to the space of angular velocities $\omega \in \mathbb{R}^3$).

We now formulate the Cartesian velocity manipulability ellipsoid associated with $J_v(\theta)$; the angular velocity manipulability ellipsoid can be formulated in an identical fashion. Assuming $J_v(\theta)$ is nonsingular at the configuration θ, $J_v(\theta)$ then maps a unit sphere in $\mathbb{R}^n$, parametrized as $\|\dot{\theta}\|^2 = 1$, to a three-dimensional ellipsoid in $\mathbb{R}^3$. The principal axes of this ellipsoid can be obtained as the eigenvectors of $J_v J_v^T \in \mathbb{R}^{3\times 3}$, with the length of each principal axis given by the corresponding eigenvalue.

A three-dimensional ellipsoid can also be drawn in the space of joint velocities as follows. First, observe that $J_v^T J_v$ is $n \times n$, but with rank 3 (as a result of our assumption that $J_v(\theta)$ is of maximal rank 3). Consequently only three of its eigenvalues are nonzero; they are, in fact, the three eigenvalues of $J_v J_v^T$. The three eigenvectors corresponding to these nonzero eigenvalues are then precisely the joint velocity vectors that map to the three principal axes of the ellipsoid in the space of Cartesian velocities.

In the absence of any preferred directions of the end-effector, one can argue that an ellipsoid that most closely resembles a sphere is the most desirable. Configurations in which the ellipsoid is spherical are called isotropic configurations, and are marked by the eigenvalues—which are proportional to the lengths of the ellipsoid's principal axes—having identical value.

6.5 Summary

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M.$$

The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector spatial velocity $\mathcal{V}_s$ via $\mathcal{V}_s = J_s(\theta)\dot{\theta}$. The i-th column of $J_s(\theta)$ is

$$\mathcal{V}_{si}(\theta) = Ad_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i), \qquad (6.43)$$

for $i = 2, \ldots, n$, with the first column $\mathcal{V}_{s1}(\theta) = \mathcal{S}_1$. $\mathcal{V}_{si}$ is the screw vector for joint i expressed in space frame coordinates, with the joint values θ assumed to be arbitrary rather than zero.

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}. \qquad (6.44)$$

The **body Jacobian** $J_b(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector spatial velocity $\mathcal{V}_b = (\omega_b, v_b)$ via

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}. \qquad (6.45)$$

The i-th column of $J_b(\theta)$ is given by

$$\mathcal{V}_{b,i}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i), \qquad (6.46)$$

for $i = n, n-1, \ldots, 2$, with $\mathcal{V}_{bn}(\theta) = \mathcal{B}_n$. $\mathcal{V}_{bi}$ is the screw vector for joint i expressed in body frame coordinates, with the joint values θ assumed to be arbitrary rather than zero. The body Jacobian is related to the space Jacobian via the relation

$$\begin{aligned} J_s(\theta) &= [\mathrm{Ad}_T sb] J_b(\theta) \\ J_b(\theta) &= [\mathrm{Ad}_T bs] J_s(\theta) \end{aligned}$$

where $T_{sb} = T$.

- Consider a force **f** applied to some point p on a rigid body. The moment **m** generated by **f** with respect to the $\{s\}$ frame origin is $\mathbf{m} = \mathbf{r} \times \mathbf{f}$, where **r** is the vector from p to the $\{s\}$ frame origin. Let $m_s \in \mathbb{R}^3$ and $f_s \in \mathbb{R}^3$ be vector representations of **m** and **f** in $\{s\}$ frame coordinates. The **spatial force in space coordinates** $\mathcal{F}_s \in \mathbb{R}^6$ is defined to be $\mathcal{F}_s = (m_s, f_s)$.

- Consider a rigid body with a body frame $\{b\}$ attached, and a force **f** applied to some point p on the rigid body. The moment **m** generated by **f** with respect to the $\{b\}$ frame origin is then $\mathbf{m} = \mathbf{r} \times \mathbf{f}$, where **r** is now the vector from p to the $\{b\}$ frame origin. Let $m_b \in \mathbb{R}^3$ and $f_b \in \mathbb{R}^3$ be vector representations of **m** and **f** in $\{b\}$ frame coordinates. The **spatial force in body coordinates** $\mathcal{F}_b \in \mathbb{R}^6$ is defined to be $\mathcal{F}_b = (m_b, f_b)$. $\mathcal{F}_b$ and $\mathcal{F}_s$ are related by

$$\begin{aligned} \mathcal{F}_b &= \mathrm{Ad}^T_{T_{sb}}(\mathcal{F}_s) = [\mathrm{Ad}_{T_{sb}}]^T \mathcal{F}_s \\ \mathcal{F}_s &= \mathrm{Ad}^T_{T_{bs}}(\mathcal{F}_b) = [\mathrm{Ad}_{T_{bs}}]^T \mathcal{F}_b. \end{aligned}$$

- Consider a spatial open chain with n one degree of freedom joints that is also assumed to be in static equilibrium. Let $\tau \in \mathbb{R}^n$ denote the vector of joint torques and forces, and $\mathcal{F} \in \mathbb{R}^6$ be the spatial force applied at the end-effector, in either space or body frame coordinates. Then τ and $\mathcal{F}$ are related by
$$\tau = J_b^T(\theta)\mathcal{F}_b = J_s^T(\theta)\mathcal{F}_s.$$

- A kinematically singular configuration for an open chain, or more simply a **kinematic singularity**, is any configuration $\theta \in \mathbb{R}^n$ at which the rank of the Jacobian (either $J_s(\theta)$ or $J_b(\theta)$) is not maximal. For spatial open chains of mobility six consisting of revolute and prismatic joints, some common singularities include (i) two collinear revolute joint axes; (ii) three coplanar and parallel revolute joint axes; (iii) four revolute joint axes intersecting at a common point; (iv) four coplanar revolute joints, and (v) six revolute joints intersecting a common line.

6.6 Notes and References

The terms spatial velocity and spatial force were first coined by Roy Featherstone [**?**], and are also referred to in the literature as twists and wrenches, respectively. There is a well developed calculus of twists and wrenches that is covered in treatments of classical screw theory, e.g., [**?**], [**?**]. Singularities of closed chains are discussed in the later chapter on closed chain kinematics. Manipulability ellipsoids and their dual, force ellipsoids, are discussed in greater detail in [**?**].

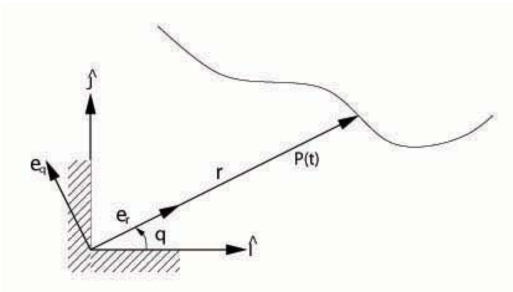

Figure 6.10: Polar coordinates.

6.7 Exercises

1. Given a particle moving in the plane, define a moving reference frame $\{\hat{e}_r, \hat{e}_\theta\}$ such that its origin is fixed to the origin of the fixed frame, and its $\hat{e}_r$ axis always points toward the particle (Figure 6.10). Let (r, θ) be the polar coordinates for the particle position, i.e., r is the distance of the particle from the origin, and θ is the angle from the horizontal line to the $\hat{e}_r$ axis. The particle position $\vec{p}$ can then be written

$$\vec{p} = r\hat{e}_r,$$

and its velocity $\vec{v}$ is given by

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r.$$

The acceleration $\vec{v}$ is the time derivative of $\vec{v}$.
(a) Express $\dot{\hat{e}}_r$ in terms of $\hat{e}_r$ and $\hat{e}_\theta$.
(b) Show that $\vec{v}$ and $\vec{a}$ are given by

$$\begin{aligned} \vec{v} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta. \end{aligned}$$

2. Let $\{\hat{I}, \hat{J}\}$ denote the unit axes of the fixed frame, and let

$$\vec{p} = X(t)\hat{I} + Y(t)\hat{J}$$

denote the position of a particle moving in the plane (see Figure 6.11). Suppose the path traced by the particle has nonzero curvature everywhere, so that for every point on the path there exists some circle tangent to the path; the center of this circle is called the **center of curvature**, while its radius is the **radius of curvature**. Clearly both the center and radius of curvature vary along the path, and are well-defined only at points of nonzero curvature (or, at points where the curvature is zero, the center of curvature can be regarded to lie at infinity).

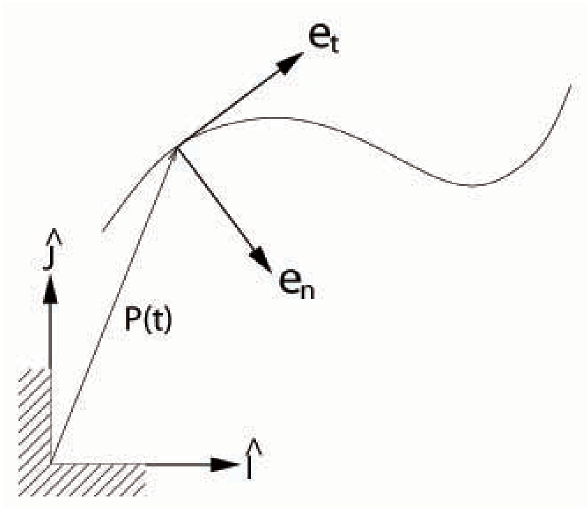

Figure 6.11: Tangential-normal coordinates.

Now attach a moving reference frame $\{\hat{e}_t, \hat{e}_n\}$ to the particle, in such a way that that $\hat{e}_t$ always points in the same direction as the velocity vector; $\hat{e}_n$ then points toward the center of curvature. Since the speed of the particle is given by

$$v = \sqrt{\dot{X}^2 + \dot{Y}^2}.$$

and $\hat{e}_t$ always points in the direction of the velocity vector, it follows that the velocity vector $\vec{v}$ of the particle can be written

$$\vec{v} = v\hat{e}_t,$$

while its acceleration is given by

$$\vec{a} = \dot{v}\hat{e}_t + v\dot{\hat{e}}_t.$$

(a) Show that $\dot{\hat{e}}_t = \|\dot{\hat{e}}_t\|\hat{e}_n$, or

$$\hat{e}_n = \frac{\dot{\hat{e}}_t}{\|\dot{\hat{e}}_t\|},$$

and that consequently the acceleration $\vec{a}$ is

$$\vec{a} = \dot{v}\hat{e}_t + v\|\dot{\hat{e}}_t\|\hat{e}_n.$$

(b) The radius of curvature ρ can be found from the following formula:

$$\begin{aligned}\rho &= \frac{v^3}{\ddot{X}\dot{Y} - \ddot{Y}\dot{X}} \\ &= \frac{(\dot{X}^2 + \dot{Y}^2)^{3/2}}{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}.\end{aligned}$$

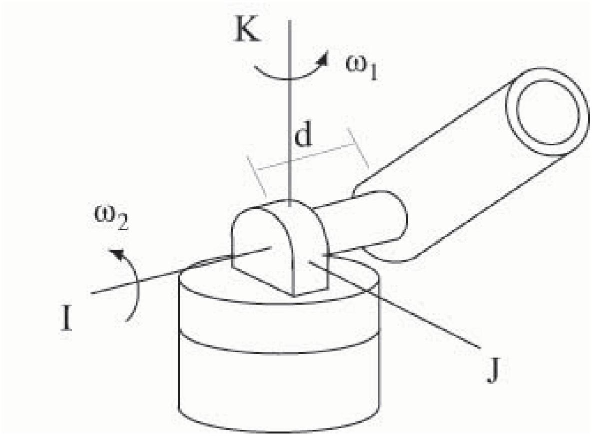

Figure 6.12: A cannon mounted on a $2R$ rotating platform.

Using the formula, show that the acceleration $\vec{a}$ can be written

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n.$$

3. In standard treatments of particle kinematics using moving frames, two moving particles, A and B, are assumed, with a moving frame $\{\hat{x}, \hat{y}, \hat{z}\}$ attached to particle A. Writing the position of particle B as

$$\vec{p}_B = \vec{p}_A + \vec{p}_{B|A},$$

where $\vec{p}_{B|A}$ denotes the vector from A to B, the following formulas for the velocity and acceleration of B are usually provided:

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \left(\dot{\vec{p}}_{B|A}\right)_{xyz} + \vec{\omega} \times \vec{p}_{B|A} \\
\vec{a}_B &= \vec{a}_A + \left(\ddot{\vec{p}}_{B|A}\right)_{xyz} + 2\vec{\omega} \times \left(\dot{\vec{p}}_{B|A}\right)_{xyz} + \vec{\alpha} \times \vec{p}_{B|A} + \vec{\omega} \times (\vec{\omega} \times \vec{p}_{B|A}),
\end{aligned}$$

where $\vec{\omega}$ and $\vec{\alpha}$ respectively denote the angular velocity and angular acceleration vector of the moving frame, and $\left(\dot{\vec{p}}_{B|A}\right)_{xyz}$ $\left(\ddot{\vec{p}}_{B|A}\right)_{xyz}$ are certain time derivatives of $\vec{p}_{B|A}$. Writing $\vec{p}_A$ and $\vec{p}_{B|A}$ in terms of fixed and moving frame coordinates, i.e.,

$$\begin{aligned}
\vec{p}_A &= X\hat{X} + Y\hat{Y} + Z\hat{Z} \\
\vec{p}_{B|A} &= x\hat{x} + y\hat{y} + z\hat{z},
\end{aligned}$$

derive the above formulas for $\vec{v}_B$ and $\vec{a}_B$. Be sure to explicitly identify all terms, in particular $\left(\dot{\vec{p}}_{B|A}\right)_{xyz}$ and $\left(\ddot{\vec{p}}_{B|A}\right)_{xyz}$.

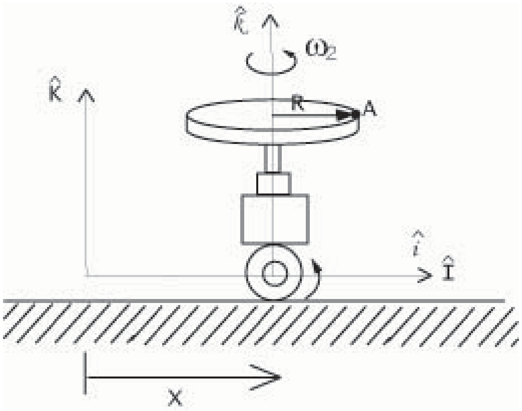

Figure 6.13: A circular plate of radius R.

4. Figure 6.12 depicts a cannon mounted on a $2R$ rotating platform at time $t = 0$. The platform rotates at constant angular velocities ω_1 and ω_2 radians/sec as shown. The axis of the cannon barrel is displaced at a distance d from the origin of the fixed frame. Assume that a cannonball is fired at $t = 0$ from the same height as the $\hat{I}$ axis, at a constant speed v_0 along the axis of the barrel.
(a) Choose a moving frame and describe how the frame moves.
(b) Determine the velocity of the cannonball at $t = 0$ in terms of the moving frame chosen in part (a).

5. As shown in Figure 6.13, a revolving circular plate of radius R, rotating at a constant angular velocity of ω_2 radians/sec, is mounted on a wheeled mobile base that moves periodically back and forth along the $\hat{I}$ axis according to

$$x(t) = a \sin \omega_1 t.$$

(a) Assuming $t = 0$ at the instant shown in the figure, determine the velocity of point A as a function of t in fixed frame coordinates.
(b) Determine the acceleration of point A as a function of t in fixed frame coordinates.

6. The circular pipe of Figure 6.14 is rotating about the $\hat{X}$ axis at a constant rate ω_1 radians/sec, while a marble D is circling the pipe at a constant speed u.
(a) Find the angle θ at which the magnitude of the velocity of D is maximal. What is the maximal velocity magnitude at this angle?
(b) Find the angle θ at which the magnitude of the acceleration of D is maximal. What is the maximal acceleration magnitude at this angle?

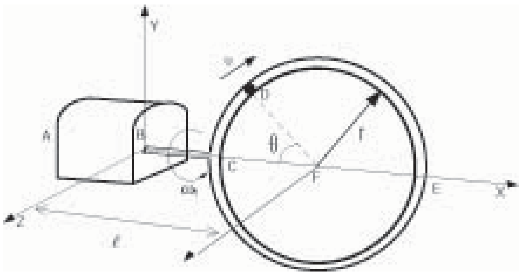

Figure 6.14: A marble traversing a rotating circular pipe.

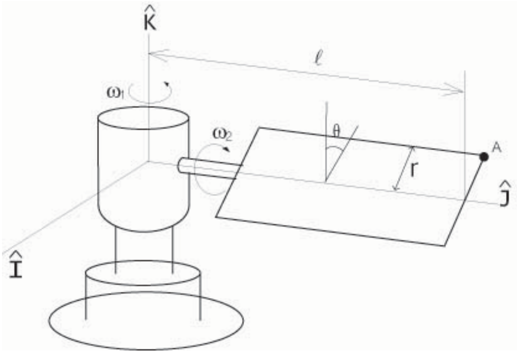

Figure 6.15: A satellite with a rotating panel.

7. The satellite of Figure 6.15 is rotating about its own vertical $\hat{K}$ axis at a constant rate ω_1 radians/sec, while at the same time its solar panel rotates at a constant rate ω_2 radians/sec as shown.
(a) Determine the velocity of point A when $\omega_1 = 0.5$, $\omega_2 = 0.25$, $l = 2m$, $r = 0.5m$, and $\theta = 30°$.
(b) Determine the acceleration of point A under the same conditions as part (a).

8. The two revolute joints in the spherical $2R$ open chain of Figure 6.16 rotating at constant angular velocities ω_1 radians/sec and ω_2 radians/sec as shown. Denote by r the length of link AB, while θ is the angle between link AB and the x-y plane.
(a) Choose a moving frame and explain how the frame moves.
(b) Determine the angular velocity and angular of link AB in terms of your moving frame coordinates chosen in part (a).
(c) Determine the velocity of point B in terms of the chosen moving frame coordinates.

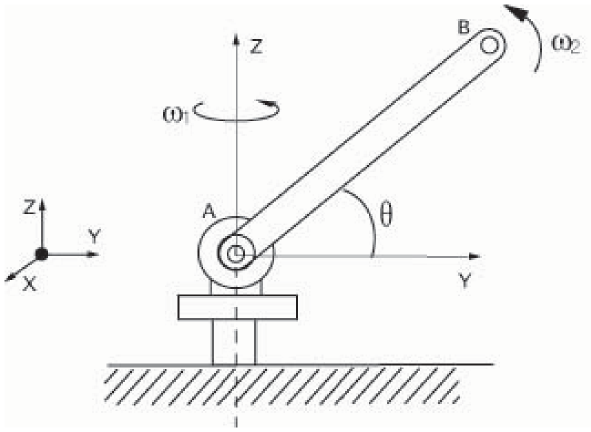

Figure 6.16: A spherical $2R$ open chain.

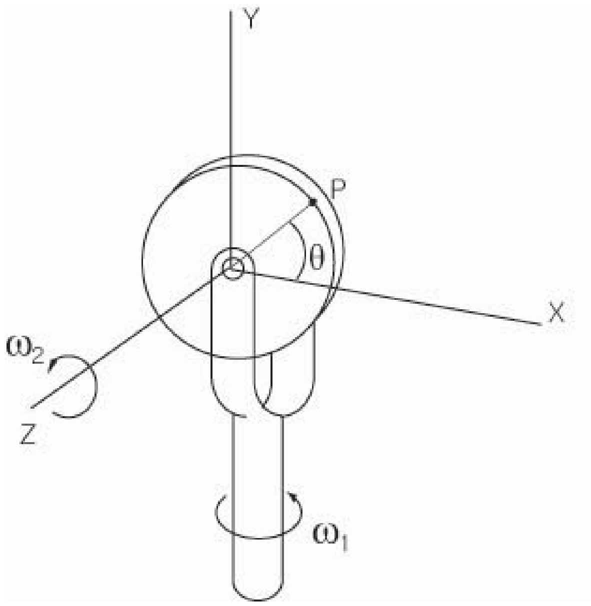

Figure 6.17: A rotating disk.

(d) Determine the acceleration of point B in terms of the chosen moving frame coordinates.
(e) Setting $\omega_1 = 0.1$, $\omega_2 = 0.2$, and $r = 100mm$, determine the velocity and acceleration of point B in terms of the fixed frame coordinates when $\theta = \pi/6$.

9. As shown in Figure 6.17, a disk of radius r spins at a constant angular

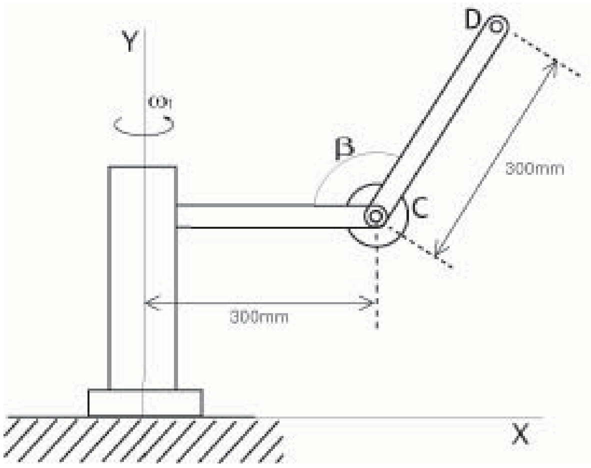

Figure 6.18: A toroidal $2R$ open chain.

velocity of ω_2 radians/sec about its horizontal axis, while at the same time the disk assembly rotates about the vertical axis at a constant angular velocity of ω_1 radians/sec.
(a) Determine the angular velocity and the angular acceleration of the disk in terms of fixed frame coordinates.
(b) Determine the velocity and the acceleration of point P as a function of the angle θ.

10. As shown in Figure 6.18, the two revolute joints of the toroidal $2R$ open chain are rotating at a constant angular velocity $\omega_1 = 0.6$ radians/sec about the $\hat{Y}$ axis, and $\omega_2 = 0.45$ radians/sec about the horizontal axis through C. When $\beta = 120°$, determine the following in terms of fixed frame coordinates:
(a) the angular acceleration of link CD.
(b) the velocity of point D.
(c) the acceleration of point D.

11. Figure 6.19 shows an RRP open chain at $t = 0$. The revolute joints rotate at constant angular velocities ω_1 and ω_2 radians/sec. Suppose the vertical position of point B is given by $x(t) = \sin t$. Determine the following quantities in terms of fixed frame coordinates.
(a) The velocity of point B at $t = 0$.
(b) The acceleration of point B at $t = 0$.

12. The square plate of Figure 6.20 rotates about axis $\hat{I}$ with angular velocity $\omega_2 = 0.5$ radians/sec and angular acceleration $\alpha_2 = 0.01$ radians/sec^2, while the circular disk attached to the square plate rotates about the axis normal

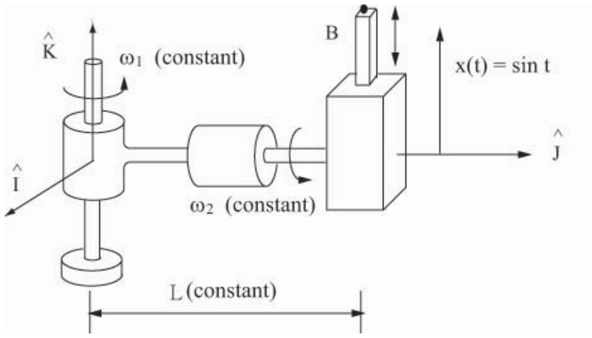

Figure 6.19: An RRP open chain.

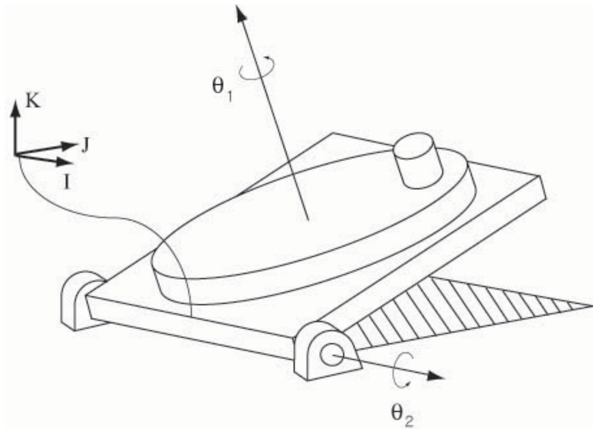

Figure 6.20: A rotating square plate.

to the plate with angular velocity $\omega_1 = 1$ radians/sec and angular acceleration $\alpha_1 = 0.5$ radians/sec^2. The radius of the circular disk is $R = 5m$, while the length of each side of the square plate is $2R = 10m$. The distance from the center of the circular disk to the small circular knob is $d = 3m$. Assume that both the disk and the square plate have zero thickness. Setting $\theta_1 = 0°$ and $\theta_2 = 45°$, find the following in terms of fixed frame coordinates:
(a) The velocity of the circular knob.
(b) The acceleration of the circular knob.

13. A person is riding the $2R$ gyro swing of Figure 6.21. Joint θ oscillates periodically according to $\theta(t) = \cos t$, and the circular plate connected to the axis of the gyro swing rotates with constant angular velocity ω_2 radians/sec. At $t = 0$, the person on the circular plate is at the maximal height as shown in the figure. Setting $l = 1$, $r = 1$, and $\omega_2 = 1$ radian/sec, determine the velocity of the person in terms of the given fixed frame coordinates when $t = \frac{\pi}{2}$.

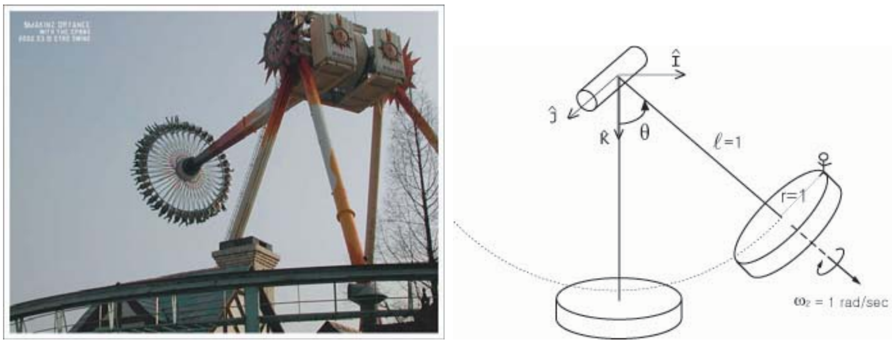

Figure 6.21: A $2R$ gyro swing.

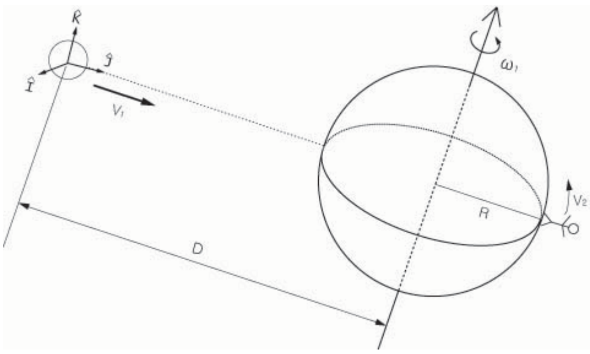

Figure 6.22: A meteorite approaching the earth.

14. As shown in Figure 6.22, a meteorite is approaching a rotating asteroid along the meteorite's $\hat{J}$ axis with velocity $v_1 = 100$ m/sec. Suppose the radius of the asteroid is $R = 1000m$, and the distance of the meteorite from the asteroid is initially $D = 10^7 m$. The asteroid takes 6 hours to complete a full revolution. An astronaut stands at the point antipodal to the expected point of collision, and unwittingly starts walking north along a longitudinal arc at a velocity of $v_2 = 1$ m/sec. After three hours, determine the velocity of the astronaut in terms of the moving frame coordinates attached to the meteorite. of the moving frame at the meteorite.

15. As shown in Figure 6.23, a clock of radius r is mounted on a gimbal assembly as shown. The angles θ_1, θ_2, and θ_3 are adjustable to arbitrary values; in the figure the angles are all assumed to be set to zero. A moving frame $\{T\}$ is attached to the tip of the clock's second hand, with its $\hat{x}$-axis aligned along the tip of the hand as shown. Setting $r = 1m$, $a = 3m$, $b = 7m$, answer the

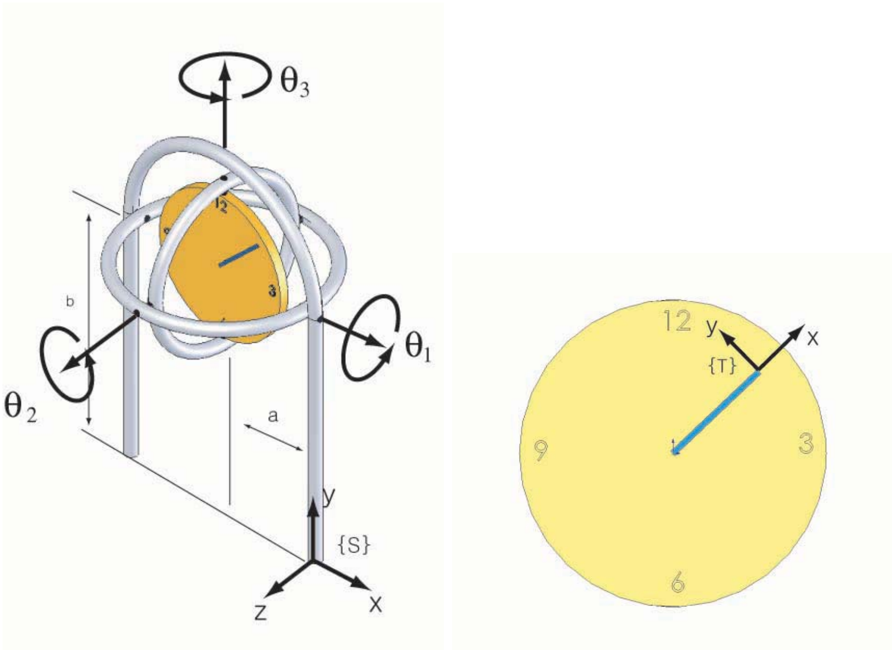

Figure 6.23: A clock mounted on a gimbal assembly.

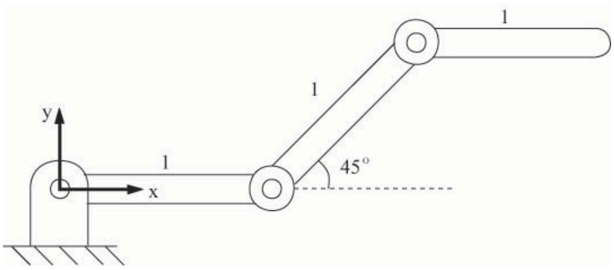

Figure 6.24: A $3R$ planar open chain.

following:
(a) Assuming the second hand starts at 12 at $t = 0$, when the second hand reaches 10, find $T_{ST} \in SE(3)$ as a function of the angles $(\theta_1, \theta_2, \theta_3)$.
(b) Setting $\theta_1 = 90°$, $\theta_2 = 0$, $\theta_3 = 90°$, find the the velocity of the tip of the second hand at the moment it passes 10.

16. The $3R$ planar open chain of Figure 6.24 is shown in its zero position.
(a) Suppose the tip must apply a force of $5N$ in the $\hat{x}$-direction. What torques should be applied at each of the joints?

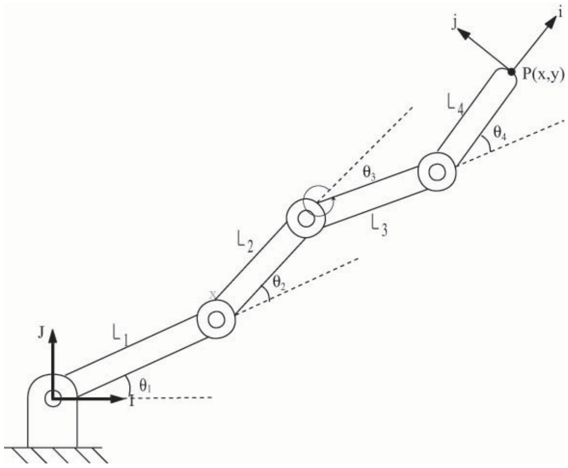

Figure 6.25: A planar $4R$ open chain.

(b) Suppose the tip must now apply a force of $5N$ in the $\hat{y}$-direction. What torques should be applied at each of the joints?

17. Answer the following questions for the $4R$ planar open chain of Figure 6.25.
(a) Derive the forward kinematics in the form

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} M.$$

where each $\mathcal{S}_\rangle \in \mathbb{R}^3$ and $M \in SE(2)$.
(b) Derive the body Jacobian.
(c) Suppose the chain is in static equilibrium at the configuration $\theta_1 = \theta_2 = 0, \theta_3 = \frac{\pi}{2}, \theta_4 = -\frac{\pi}{2}$, and a force $f = (10, 10, 0)$ and moment $m = (0, 0, 10)$ are applied to the tip (both f and m are expressed with respect to the fixed frame). What are the torques experienced at each of the joints?
(d) Under the same conditions as (c), suppose that a force $f = (-10, 10, 0)$ and moment $m = (0, 0, -10)$ are applied to the tip. What are the torques experienced at each of the joints?
(e) Find all kinematic singularities for this chain.

18. Referring to Figure 6.26, the rigid body rotates about the point (L, L) with angular velocity $\dot{\theta} = 1$.
(a) Find the position of point P on the moving body with respect to the fixed reference frame in terms of θ.
(b) Find the velocity of point P in terms of the fixed frame.
(c) What is T_{fb}, the displacement of frame $\{b\}$ as seen from the fixed frame $\{f\}$?
(d) Find the spatial velocity of T_{fb} in body coordinates.

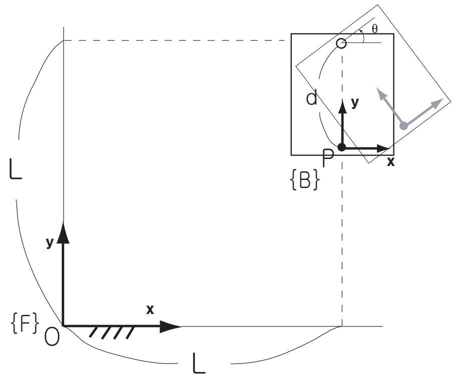

Figure 6.26: A rigid body rotating in the plane.

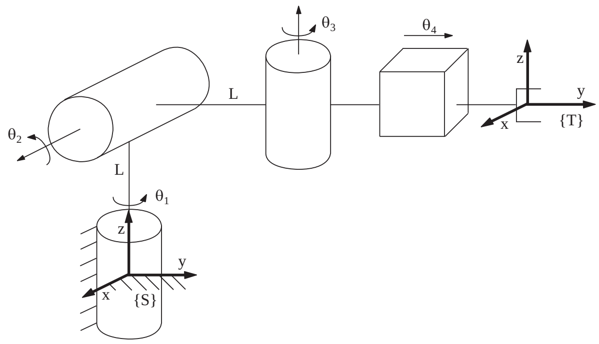

Figure 6.27: An *RRRP* spatial open chain.

(e) Find the spatial velocity of T_{fb} in space coordinates.
(f) What is the relation between the spatial velocities obtained in (d) and (e)?
(g) What is the relation between the spatial velocity obtained in (d) and $\dot{P}$ obtained in (b)?
(h) What is the relation between the spatial velocity obtained in (e) and $\dot{P}$ obtained in (b)?

19. The *RRRP* chain of Figure 6.27 is shown in its zero position.
(a) Determine the body Jacobian $J_b(\theta)$ when $\theta_1 = \theta_2 = 0, \theta_3 = \pi/2, \theta_4 = L$.
(b) Determine the linear velocity of the end-effector frame, in fixed frame coordinates, when $\theta_1 = \theta_2 = 0, \theta_3 = \pi/2, \theta_4 = L$ and $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}_4 = 1$.

20. The spatial $3R$ open chain of Figure 6.28 is shown in its zero position.
(a) In its zero position, suppose we wish to make the end-effector move with linear velocity $v_{tip} = (10, 0, 0)$, where v_{tip} is expressed with respect to the space frame $\{s\}$. What are the necessary input joint velocities $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$?

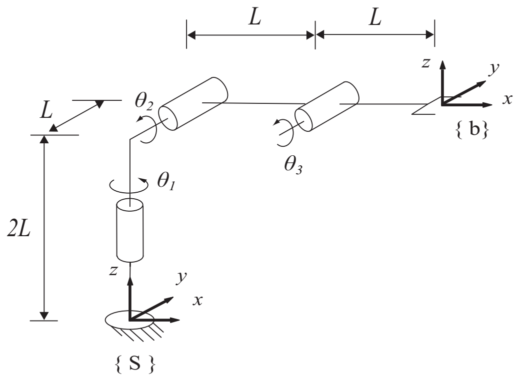

Figure 6.28: A spatial $3R$ open chain.

(b) Suppose the robot is in the configuration $\theta_1 = 0, \theta_2 = 45°, \theta_3 = -45°$. Assuming static equilibrium, suppose we wish to generate an end-effector force $f_b = (10, 0, 0)$, where f_b is expressed with respect to the end-effector frame $\{b\}$. What are the necessary input joint torques τ_1, τ_2, τ_3?
(c) Under the same conditions as in (b), suppose we now seek to generate an end-effector moment $m_b = (10, 0, 0)$, where m_b is expressed with respect to the end-effector frame {b}. What are the necessary input joint torques τ_1, τ_2, τ_3?
(d) Suppose the maximum allowable torques for each joint motor are

$$\|\tau_1\| \leq 10, \|\tau_2\| \leq 20, \|\tau_3\| \leq 5.$$

In the home position, what is the maximum force that can be applied by the tip in the end-effector frame x-direction?

21. The spatial $PRRRRP$ open chain of Figure 6.29 is shown in its zero position.
(a) At the zero position, find the first three columns of the space Jacobian.
(b) Find all configurations at which the first three columns of the space Jacobian become linearly dependent.
(c) Suppose the chain is in the configuration $\theta_1 = \theta_2 = \theta_3 = \theta_5 = \theta_6 = 0, \theta_4 = 90^o$. Assuming static equilibrium, suppose a pure force $f_b = (10, 0, 10)$ is applied to the origin of the end-effector frame, where f_b is expressed in terms of the end-effector frame. Find the joint torques τ_1, τ_2, τ_3 experienced at the first three joints.

22. Consider the $PRRRRR$ spatial open chain of Figure 6.30 shown in its zero position. The distance from the origin of the fixed frame to the origin of the end-effector frame at the home position is L.
(a) Determine the first three columns of the space Jacobian J_s.
(b) Determine the last two columns of the body Jacobian J_b.

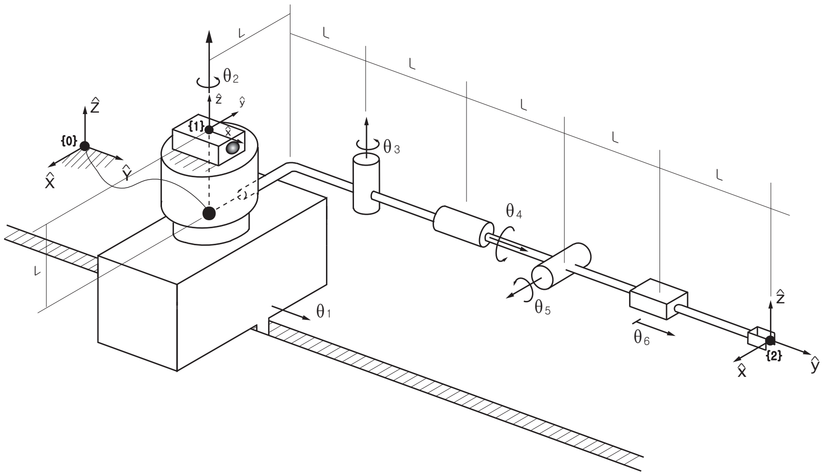

Figure 6.29: A spatial $PRRRRP$ open chain.

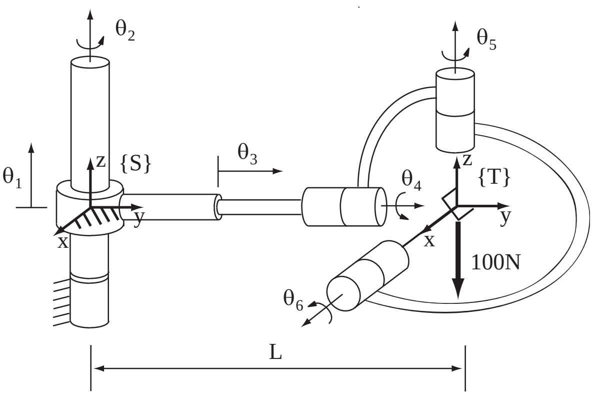

Figure 6.30: A $PRRRRR$ spatial open chain.

(c) For what value of L is the home position a singularity?
(d) In the zero position, what joint torques must be applied in order to generate a pure end-effector force of $100N$ in the -$\hat{z}$ direction?

23. Find all kinematic singularities of the $3R$ wrist with the following forward kinematics:

$$R = e^{[\omega_1]\theta_1} e^{[\omega_2]\theta_2} e^{[\omega_3]\theta_3}$$

where $\omega_1 = (0,0,1)$, $\omega_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})$, and $\omega_3 = (1,0,0)$.

24. Show that a six degree of freedom spatial open chain is in a kinematic singularity when any two of its revolute joint axes are parallel, and any prismatic

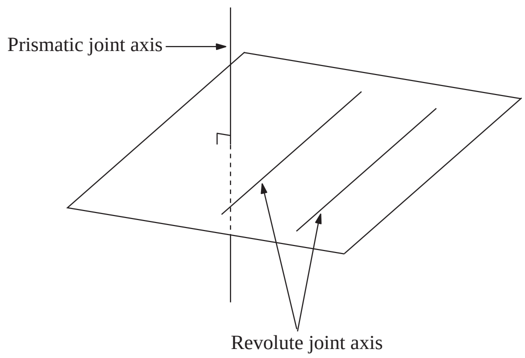

Figure 6.31: A kinematic singularity involving prismatic and revolute joints.

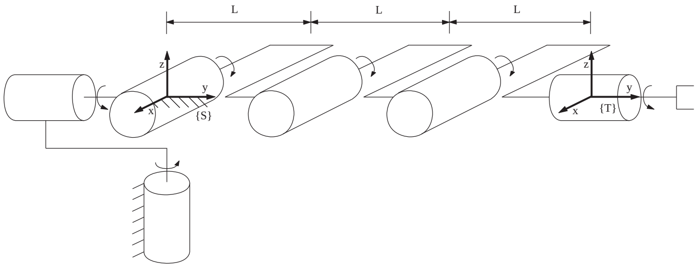

Figure 6.32: Singularities of an elbow-type $6R$ open chain.

joint axis is normal to the plane spanned by the two parallel revolute joint axes (see Figure 6.31).

25. (a) Determine the space Jacobian $J_s(\theta)$ of the $6R$ spatial open chain of Figure 6.32.
(b) Find the kinematic singularities of the given chain. Explain each singularity in terms of the alignment of the joint screws, and the directions in which the end-effector loses one or more degrees of freedom of motion.

26. The spatial $PRRRRP$ open chain of Figure 6.33 is shown in its zero position.
(a) Determine the first 4 columns of the space Jacobian $J_s(\theta)$.
(b) Determine whether the zero position is a kinematic singularity.
(c) Calculate the joint torques required for the tip to apply the following end-effector spatial forces:

(i) $F_s = (0, 1, -1, 1, 0, 0)^T$

(ii) $F_s = (1, -1, 0, 1, 0, -1)^T$.

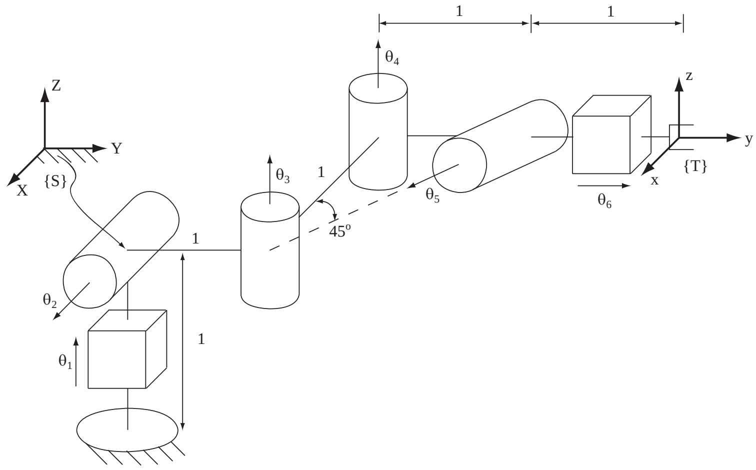

Figure 6.33: A spatial $PRRRRP$ open chain with a skewed joint axis.

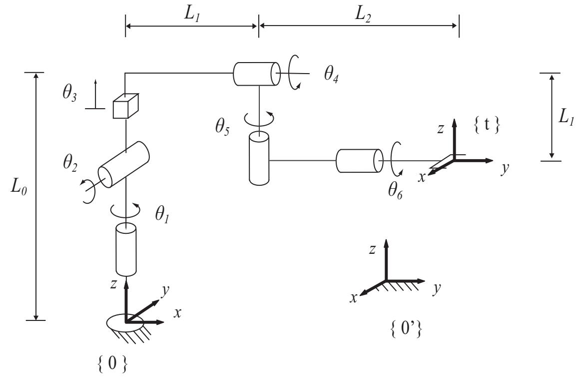

Figure 6.34: A spatial $RRPRRR$ open chain.

27. The spatial $RRPRRR$ open chain of Figure 6.34 is shown in its zero position.
(a) For the fixed frame {0} and tool frame {t} as shown, express the forward kinematics in the following product of exponentials form:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M.$$

(b) Find the first three columns of the space Jacobian $J_s(\theta)$.
(c) Suppose that the fixed frame {0} is moved to another location {0′} as shown in the figure. Find the first three columns of the space Jacobian $J_s(\theta)$ with respect to this new fixed frame.

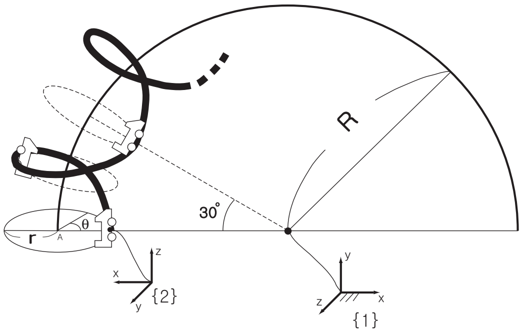

Figure 6.35: A rollercoaster undergoing a screw motion.

(d) Determine if the zero position is a kinematic singularity, and if so, provide a geometric description in terms of the joint screw axes.

28. The rollercoaster of Figure 6.35 undergoes a screw motion as shown: point A traces a circle of radius R, and the rollercoaster moves in screw-like fashion at a distance r from this larger circle. The roller coaster completes one revolution about this larger circle when point A traverses 30° along the larger circle.
(a) Find T_{12}, the relative displacement of the rollercoaster frame $\{2\}$ as seen from the fixed frame $\{1\}$, in terms of the angle θ as indicated in the figure.
(b) Derive the space Jacobian for $T_{12}(\theta)$.

29. Two frames $\{a\}$ and $\{b\}$ are attached to a moving rigid body. Show that the spatial velocity of $\{a\}$ in space frame coordinates is the same as the spatial velocity of $\{b\}$ in space frame coordinates.

30. Consider an n-link open chain, with reference frames attached to each link. Let

$$T_{0k} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_k]\theta_k} M_k, \quad k = 1, \ldots, n$$

be the forward kinematics up to link frame $\{k\}$. Let $J_s(\theta)$ be the space Jacobian for T_{0n}. The columns of $J_s(\theta)$are denoted

$$J_s(\theta) = \left[\begin{array}{ccc} \mathcal{V}_{s1}(\theta) & \cdots & \mathcal{V}_{sn}(\theta) \end{array} \right].$$

Let $[\mathcal{V}_k] = \dot{T_{0k}} T_{0k}^{-1}$ be the spatial velocity of link frame $\{k\}$ in frame $\{k\}$ coordinates.
(a) Derive explicit expressions for $\mathcal{V}_2$ and $\mathcal{V}_3$.
(b) Based on your results from (a), derive a recursive formula for $\mathcal{V}_{k+1}$ in terms of $\mathcal{V}_k$, $\mathcal{V}_{s1}, \ldots, \mathcal{V}_{s,k+1}$, and $\dot{\theta}$.