

Chapter 5

Forward Kinematics

The **forward kinematics** of a robot refers to the calculation of the position and orientation of its end-effector frame from its joint values. Figure 5.1 illustrates the forward kinematics problem for the 3R planar open chain. Starting from the base link, the link lengths are L_1 , L_2 , and L_3 . Choose a fixed frame $\{0\}$ with origin located at the base joint as shown, and assume an end-effector frame $\{4\}$ has been attached to the tip of the third link. The Cartesian position (x, y) and orientation ϕ of the end-effector frame as a function of the joint angles $(\theta_1, \theta_2, \theta_3)$ are then given by

$$x = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \quad (5.1)$$

$$y = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \quad (5.2)$$

$$\phi = \theta_1 + \theta_2 + \theta_3. \quad (5.3)$$

If one is only interested in the (x, y) position of the end-effector, the robot's task space is then taken to be the x - y plane, and the forward kinematics would then consist of Equations (5.1)-(5.2) only. If the end-effector's position and orientation both matter, the forward kinematics would then consist of the three equations (5.1)-(5.3).

While the above analysis can be done using only basic trigonometry, it is not difficult to imagine that for more general spatial chains, the analysis can become considerably more complicated. A more systematic method of deriving the forward kinematics would be to first attach reference frames to each of the links; in Figure 5.1 the three link reference frames are respectively labelled $\{1\}$, $\{2\}$, and $\{3\}$. The forward kinematics can then be written as a product of four $SE(2)$ matrices:

$$T_{04} = T_{01}T_{12}T_{23}T_{34},$$

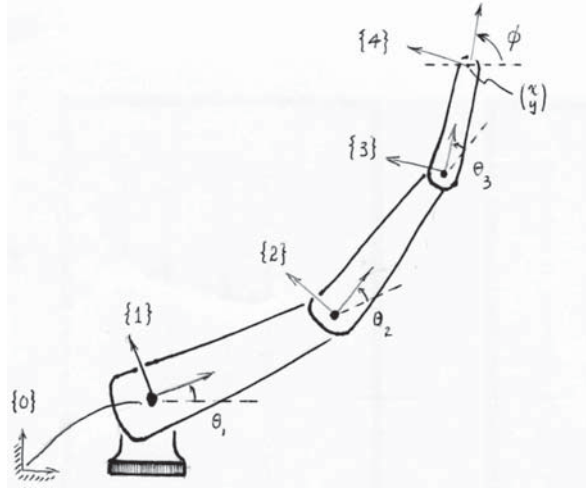


Figure 5.1: Forward kinematics of a 3R planar open chain.

where

$$\begin{aligned}
 T_{01} &= \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & T_{12} &= \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & L_1 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 T_{23} &= \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & L_2 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & T_{34} &= \begin{bmatrix} 1 & 0 & L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
 \end{aligned} \quad (5.4)$$

Observe that T_{34} is constant, and that each remaining $T_{i-1,i}$ depends only on the joint variable θ_i .

Alternatively, if we regard each joint axis as being the axis of a zero-pitch screw motion, then observe that joint axis three can be thought of as applying a screw motion to link three. Assuming joints θ_1 and θ_2 are both held constant at zero, from the matrix exponential representation for screw motions covered in the previous chapter, we can write

$$T_{04} = e^{[S_3]\theta_3} M, \quad (5.5)$$

where

$$M = \begin{bmatrix} 1 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (5.6)$$

describes the position and orientation of frame $\{4\}$ when all joints are set to zero, and

$$[S_3] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 \end{bmatrix}. \quad (5.7)$$

Similarly, joint axis two can be viewed as applying a screw motion to the link two-link three pair; assuming joint θ_1 is held constant at zero, we can write

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M, \quad (5.8)$$

where $[\mathcal{S}_3]$ and M are as defined previously, and

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -L_1 \\ 0 & 0 & 0 \end{bmatrix}. \quad (5.9)$$

Finally, joint axis one can be viewed as applying a screw motion to the entire three-link assembly; for arbitrary values of $(\theta_1, \theta_2, \theta_3)$ we can therefore write

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M, \quad (5.10)$$

where $[\mathcal{S}_2]$, $[\mathcal{S}_3]$, and M are as defined previously, and

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (5.11)$$

The forward kinematics can thus be expressed as a product of matrix exponentials, each corresponding to a screw motion. Note that this latter derivation of the forward kinematics does not make use of any link reference frames.

In this chapter we shall consider the forward kinematics of general open chains, taking the task space to be the position and orientation of the end-effector frame in the most general case. Two widely used forward kinematic representations for open chains will be considered: the homogeneous transformation representation based on the **Denavit-Hartenberg** (D-H) parameters, which corresponds to Equation (5.4), and the screw-theoretic formulation based on the **Product of Exponentials** (PoE) formula, which corresponds to Equation (5.11). The advantage of the D-H representation is that it is a minimal representation, in the sense of requiring the smallest number of parameters to describe the robot's kinematic structure. The PoE representation is not minimal, but as we show later, has so many other advantages over the D-H representation (e.g., no link frames are necessary) that with few exceptions it will be our preferred choice of forward kinematics representation. We include the D-H representation mostly for completeness.

5.1 Denavit-Hartenberg Representation

The basic idea underlying the Denavit-Hartenberg approach to forward kinematics is to attach reference frames to each link of the open chain, and to derive the forward kinematics based on knowledge of the relative displacements between adjacent link frames. Assume that a fixed reference frame has been established, and that a reference frame (the end-effector frame) has been attached to some

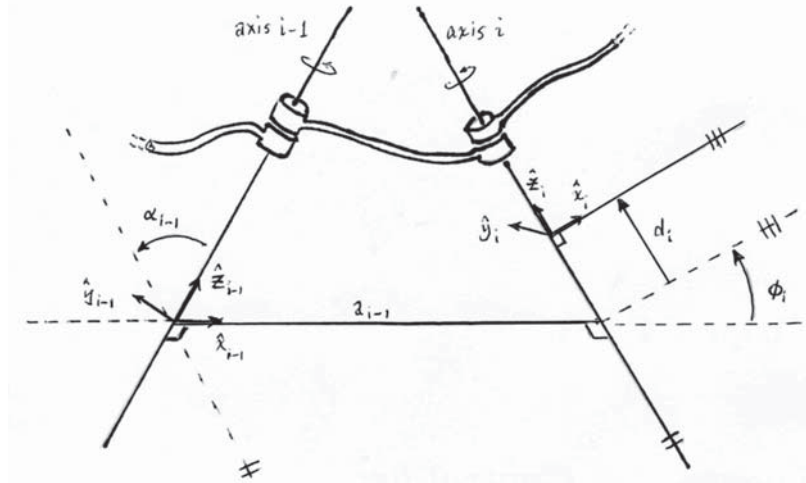


Figure 5.2: Illustration of Denavit-Hartenberg parameters.

point on the last link of the open chain. For a chain consisting of n one degree of freedom joints, the links are numbered sequentially from 0 to n , in which the ground link is labelled 0, and the end-effector frame is attached to some point on link n . Reference frames attached to the links are also correspondingly labelled from $\{0\}$ (the fixed frame) to $\{n\}$ (the end-effector frame). The joint variable corresponding to the i -th joint is denoted θ_i . The forward kinematics of the n -link open chain can then be expressed as

$$T_{0n}(\theta_1, \dots, \theta_n) = T_{01}(\theta_1)T_{12}(\theta_2) \cdots T_{n-1,n}(\theta_n), \quad (5.12)$$

where $T_{i,i-1} \in SE(3)$ denotes the relative displacement between link frames $\{i-1\}$ and $\{i\}$. Depending on how the link reference frames have been chosen, for open chains each $T_{i-1,i}$ can be obtained in a straightforward fashion.

5.1.1 Assigning Link Frames

Rather than attaching reference frames to each link in some arbitrary fashion, in the Denavit-Hartenberg convention a set of rules for assigning link frames is observed. Figure 5.2 illustrates the frame assignment convention for two adjacent revolute joints $i-1$ and i that are connected by link $i-1$.

The first rule is that the \hat{z} -axis coincides with joint axis i , and \hat{z}_{i-1} coincides with joint axis $i-1$. The direction of each link frame \hat{z} -axis is determined via the right-hand rule, i.e., such that positive rotations are counterclockwise about the \hat{z} -axis.

Once the \hat{z} -axis direction has been assigned, the next rule determines the origin of the link reference frame. First, find the line segment that orthogonally intersects both joint axes \hat{z}_{i-1} and \hat{z}_i . For now let us assume that this line

segment is unique; the case where it is not unique (i.e., when the two joint axes are parallel), or fails to exist (i.e., when the two joint axes intersect), is addressed later. Connecting joint axes $i - 1$ and i by a mutually perpendicular line, the origin of frame $\{i - 1\}$ is then located at the point where this line intersects joint axis $i - 1$.

Determining the remaining \hat{x} - and \hat{y} -axes of each link reference frame is now straightforward: the \hat{x} axis is chosen to be in the direction of the mutually perpendicular line pointing from the $i - 1$ axis to the i axis. The \hat{y} -axis is then uniquely determined from the cross-product $\hat{x} \times \hat{y} = \hat{z}$. Figure 5.2 depicts the link frames i and $i - 1$ chosen according to this convention.

Having assigned reference frames in this fashion for links i and $i - 1$, we now define four parameters that exactly specify $T_{i-1,i}$:

- The length of the mutually perpendicular line, denoted by the scalar a_{i-1} , is called the **link length** of link $i - 1$. Despite its name, this link length does not necessarily correspond to the actual length of the physical link.
- The **link twist** α_{i-1} is the angle from \hat{z}_{i-1} to \hat{z}_i , measured about \hat{x}_{i-1} .
- The **link offset** d_i is the distance from the intersection of \hat{x}_{i-1} and \hat{z}_i to the link i frame origin (the positive direction is defined to be along the \hat{z}_i axis).
- The **joint angle** ϕ_i is the angle from \hat{x}_{i-1} to \hat{x}_i , measured about the \hat{z}_i -axis in the right-hand sense.

These parameters constitute the Denavit-Hartenberg parameters. For an open chain with n one degree-of-freedom joints, the $4n$ Denavit-Hartenberg parameters are sufficient to completely describe the forward kinematics. In the case of an open chain with all joints revolute, the link lengths a_{i-1} , twists α_{i-1} , and offset parameters d_i are all constant, while the joint angle parameters ϕ_i act as the joint variables.

We now consider the case where the mutually perpendicular line is undefined or fails to be unique, as well as when some of the joints are prismatic, and finally, how to choose the ground and end-effector frames.

When Adjacent Revolute Joint Axes Intersect

If two adjacent revolute joint axes intersect each other, then the mutually perpendicular line between the joint axes fails to exist. In this case the link length is set to zero, and we choose \hat{x}_{i-1} to be perpendicular to the plane spanned by \hat{z}_{i-1} and \hat{z}_i . There are two possibilities here, both of which are acceptable: one leads to a positive value of the twist angle α_{i-1} , while the other leads to a negative value.

When Adjacent Revolute Joint Axes are Parallel

The second special case occurs when two adjacent revolute joint axes are parallel. In this case there exist many possibilities for a mutually perpendicular

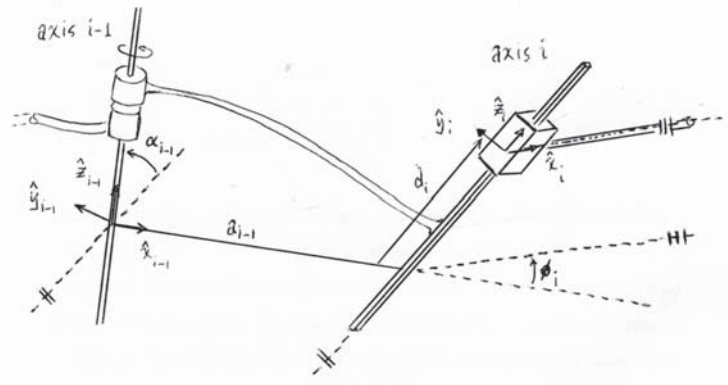


Figure 5.3: Link frame assignment convention for prismatic joints. Joint $i - 1$ is a revolute joint, while joint i is a prismatic joint.

line, all of which are valid (more precisely, a one-parameter family of mutual perpendicular lines is said to exist). Again, it is important to detail precisely how the link frames are assigned. A useful guide is to try to choose the mutually perpendicular line that is the most physically intuitive, and simplifies as many Denavit-Hartenberg parameters as possible (e.g., such that their values become zero).

Prismatic Joints

For prismatic joints, the \hat{z} -direction of the link reference frame is chosen to be along the positive direction of translation. This convention is consistent with that for revolute joints, in which the \hat{z} -axis indicates the positive axis of rotation. With this choice the link offset parameter d_i now becomes the joint variable (see Figure 5.3). The procedure for choosing the link frame origin, as well as the remaining \hat{x} and \hat{y} -axes, remains the same as for revolute joints.

Assigning the Ground and End-Effector Frames

Our frame assignment procedure described thus far does not specify how to choose the ground and final link frames. Here as before, a useful guideline is to choose initial and final frames that are the most physically intuitive, and simplify as many Denavit-Hartenberg parameters as possible. This usually implies that the ground frame is chosen to coincide with the link 1 frame in its zero (rest) position; in the event that the joint is revolute, this choice forces $a_0 = \alpha_0 = d_1 = 0$, while for a prismatic joint we have $a_0 = \alpha_0 = \phi_1 = 0$. The end-effector frame is typically attached to some reference point on the end-effector, usually at a location that makes the description of the task intuitive and natural, and also simplifies as many of the Denavit-Hartenberg parameters as possible (e.g., their values become zero).

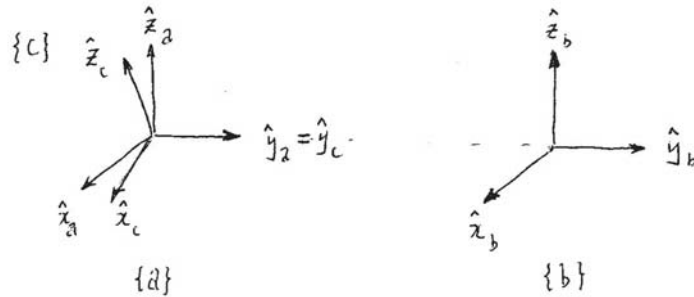


Figure 5.4: An example of three frames $\{a\}$, $\{b\}$, and $\{c\}$, in which the transformations T_{ab} and T_{ac} cannot be described by any set of Denavit-Hartenberg parameters.

It is important to realize that arbitrary choices of the ground and end-effector frames may not always be possible, since there may not exist a valid set of Denavit-Hartenberg parameters to describe the relative transformation; we elaborate on this point below.

5.1.2 Why Four Parameters are Sufficient

In our earlier study of spatial displacements, we argued that a minimum of six independent parameters were required to describe the relative displacement between two frames in space: three for the orientation, and three for the position. Based on this result, it would seem that for an n -link arm, a total of $6n$ parameters would be required to completely describe the forward kinematics (each $T_{i-1,i}$ in the above equation would require six parameters). Surprisingly, in the Denavit-Hartenberg parameter representation only four parameters are required for each transformation $T_{i-1,i}$. Although this result may at first appear to contradict our earlier results, this reduction in the number of parameters is accomplished by the carefully stipulated rules for assigning link reference frames. If the link reference frames had been assigned in arbitrary fashion, then more parameters would have been required.

Consider, for example, the link frames shown in Figure 5.4. The transformation from frame $\{a\}$ to frame $\{b\}$ is a pure translation along the \hat{y} -axis of frame $\{a\}$. If one were to try to express the transformation T_{ab} in terms of the Denavit-Hartenberg parameters (α, a, d, θ) as prescribed above, it should become apparent that no such set of parameter values exist. Similarly, the transformation T_{ac} also does not admit a description in terms of Denavit-Hartenberg parameters, as only rotations about the \hat{x} - and \hat{z} - axes are permissible. Under our Denavit-Hartenberg convention, only rotations and translations along the \hat{x} and \hat{z} axes are allowed, and no combination of such motions can achieve the transformations shown in Figure 5.4.

Given that the Denavit-Hartenberg convention uses exactly four parameters to describe the transformation between link frames, one might naturally wonder if the number of parameters can be reduced even further, by an even more clever set of link frame assignment rules. Denavit and Hartenberg show that this is not possible, and that four is the minimum number of parameters [7].

We end this section with a reminder that there are alternative conventions for assigning link frames. Whereas we chose the \hat{z} -axis to coincide with the joint axis, some authors choose the \hat{x} -axis, and reserve the \hat{z} -axis to be the direction of the mutually perpendicular line. To avoid ambiguities in the interpretation of the Denavit-Hartenberg parameters, it is essential to include a concise description of the link frames together with the parameter values.

5.1.3 Manipulator Forward Kinematics

Once all the transformations $T_{i-1,i}$ between adjacent link frames are known in terms of their Denavit-Hartenberg parameters, the forward kinematics is obtained by sequentially multiplying these link transformations. Each link frame transformation is of the form

$$\begin{aligned} T_{i-1,i} &= \text{Rot}(\hat{x}, \alpha_{i-1}) \cdot \text{Trans}(\hat{x}, a_{i-1}) \cdot \text{Trans}(\hat{z}, d_i) \cdot \text{Rot}(\hat{z}, \phi_i) \\ &= \begin{bmatrix} \cos \phi_i & -\sin \phi_i & 0 & a_{i-1} \\ \sin \phi_i \cos \alpha_{i-1} & \cos \phi_i \cos \alpha_{i-1} & -\sin \alpha_{i-1} & -d_i \sin \alpha_{i-1} \\ \sin \phi_i \sin \alpha_{i-1} & \cos \phi_i \sin \alpha_{i-1} & \cos \alpha_{i-1} & d_i \cos \alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

where

$$\text{Rot}(\hat{x}, \alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha_{i-1} & -\sin \alpha_{i-1} & 0 \\ 0 & \sin \alpha_{i-1} & \cos \alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.13)$$

$$\text{Trans}(\hat{x}, a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.14)$$

$$\text{Trans}(\hat{z}, d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.15)$$

$$\text{Rot}(\hat{z}, \phi_i) = \begin{bmatrix} \cos \phi_{i-1} & -\sin \phi_{i-1} & 0 & 0 \\ -\sin \phi_{i-1} & \cos \phi_{i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5.16)$$

A useful way to visualize $T_{i,i-1}$ is to transport frame $\{i-1\}$ to frame $\{i\}$ via the following sequence of four transformations:

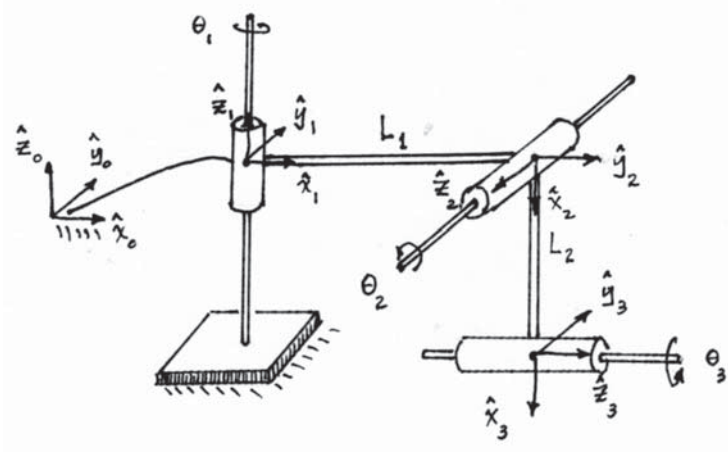


Figure 5.5: A 3R spatial open chain.

- (i) Rotate frame $\{i - 1\}$ about its \hat{x} axis by an angle α_{i-1} .
- (ii) Translate this new frame along its \hat{x} axis by a distance a_{i-1} .
- (iii) Translate this new frame along its \hat{z} axis by a distance d_i .
- (iv) Rotate this new frame about its \hat{z} axis by an angle ϕ_i .

Note that switching the order of the first and second steps will not change the final form of $T_{i-1,i}$. Similarly, the order of the third and fourth steps can also be switched without affecting $T_{i-1,i}$.

5.1.4 Examples

We now derive the Denavit-Hartenberg parameters for some common spatial open chain structures.

Example: A 3R Spatial Open Chain

Consider the 3R spatial open chain of Figure 5.5, shown in its zero position (i.e., with all its joint variables set to zero). The assigned link reference frames are shown in the figure, and the corresponding Denavit-Hartenberg parameters listed in the following table:

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	L_1	0	$\theta_2 - 90^\circ$
3	-90°	L_2	0	θ_3

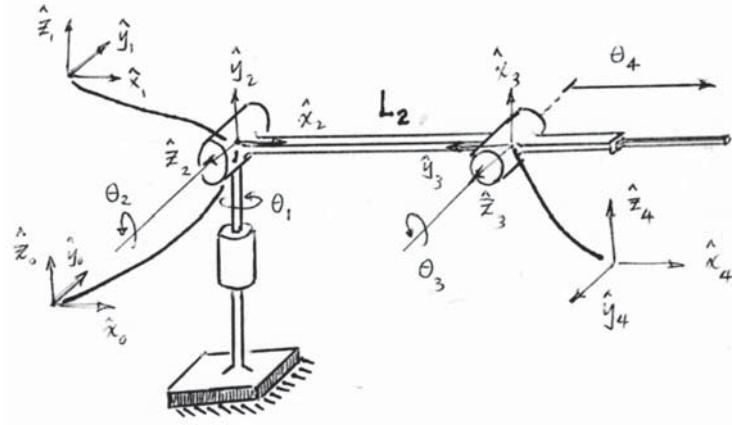


Figure 5.6: An RRRP spatial open chain.

Note that frames $\{1\}$ and $\{2\}$ are uniquely specified from our frame assignment convention, but that we have some latitude in choosing frames $\{0\}$ and $\{3\}$. Here we choose the ground frame $\{0\}$ to coincide with frame $\{1\}$ (resulting in $\alpha_0 = a_0 = d_1 = 0$), and frame $\{3\}$ such that $\hat{x}_3 = \hat{x}_2$ (resulting in no offset to the joint angle θ_3).

Example: A Spatial RRRP Open Chain

The next example we consider is the four degree-of-freedom RRRP spatial open chain of Figure 5.6, here shown in its zero position. The link frame assignments are as shown, and the corresponding Denavit-Hartenberg parameters are listed in the following table:

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	0	0	θ_2
3	0	L_2	0	$\theta_3 + 90^\circ$
4	90°	0	θ_4	0

The four joint variables are $(\theta_1, \theta_2, \theta_3, \theta_4)$, where θ_4 is the displacement of the prismatic joint. As in the previous example, the ground frame $\{0\}$ and final link frame $\{4\}$ have been chosen to make as many of the Denavit-Hartenberg parameters zero.

Example: A Spatial 6R Open Chain

The final example we consider is a widely used six 6R robot arm (Figure 5.7). This open chain has six rotational joints: the first three joints function as a Cartesian positioning device, while the last three joints act as a ZYZ type

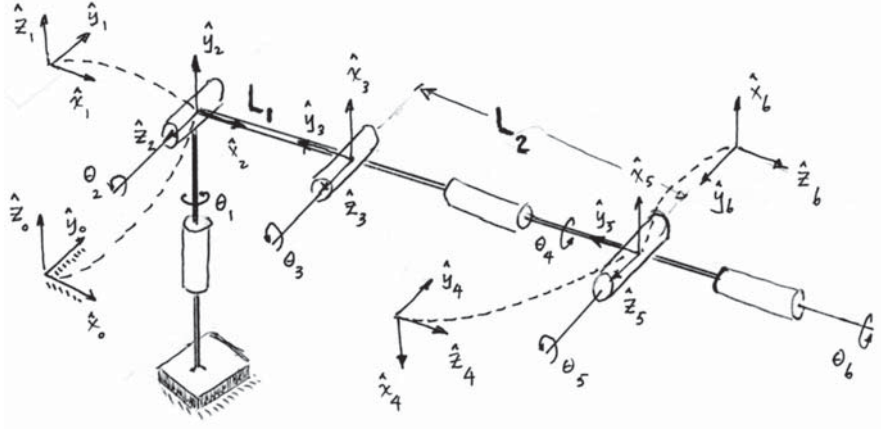


Figure 5.7: A 6R spatial open chain.

wrist. The link frames are shown in the figure, and the corresponding Denavit-Hartenberg parameters are listed in the following table:

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	0	0	θ_2
3	0	L_1	0	$\theta_3 + 90^\circ$
4	90°	0	L_2	$\theta_4 + 180^\circ$
5	90°	0	0	$\theta_5 + 180^\circ$
6	90°	0	0	θ_6

5.2 Product of Exponentials Formula

In this section we present an alternative representation of the forward kinematics, called the **product of exponentials** (PoE) formula, that is based on the matrix exponential representation for screw motions introduced in the previous chapter. We begin by recalling the Chasles-Mozzi theorem: Any displacement of a rigid body can be expressed as a finite rotation about some fixed line in space (the screw axis), followed by a finite translation parallel to the fixed line. Since both the rotation and translation are taken about the same fixed line, reversing their order still results in the same displacement.

To state things more precisely, assume a fixed frame has been chosen, and a body-fixed frame attached to the rigid body. If the body is displaced from some initial configuration $M \in SE(3)$ to another configuration $T \in SE(3)$, the displacement can then be expressed as

$$T = e^{[S]\theta} M, \quad [S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}, \quad (5.17)$$

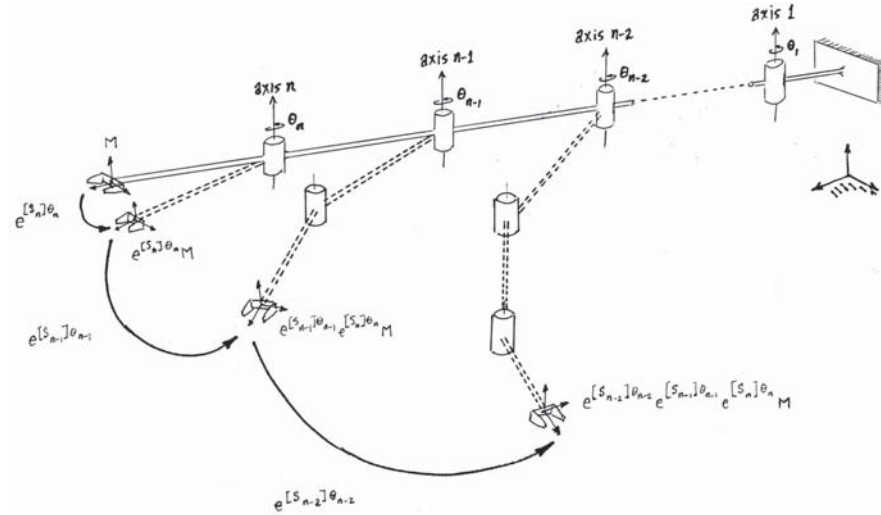


Figure 5.8: Illustration of the PoE formula for an n -link spatial open chain.

where $\mathcal{S} = (\omega, v)$ denotes the six-dimensional twist representation of the screw motion and $[S]$ its 4×4 matrix representation, and $\theta \in \mathbb{R}$ is the joint displacement. Recall that for displacements that are not pure translations, $\omega \in \mathbb{R}^3$ is a unit vector in the direction of the screw axis and $[\omega] \in \mathbb{R}^{3 \times 3}$ its skew-symmetric matrix representation, and $v = -\omega \times q + h\omega$, where $q \in \mathbb{R}^3$ is a point on the screw axis and $h \in \mathbb{R}$ is the screw pitch. If the displacement is a pure translation, ω is zero, and $v \in \mathbb{R}^3$ is a unit vector in the direction of positive translation.

The two limiting cases for a general screw motion are a pure rotation (corresponding to zero pitch $h = 0$) and a pure translation (corresponding to infinite pitch $h = \infty$). These two motions also happen to correspond to the two most prevalent joint types found in robots, the revolute joint and prismatic joint. The PoE formula takes advantage of this identification, by expressing the motions of revolute and prismatic joints as screw motions. We now derive the PoE formula for general spatial open chains.

5.2.1 First Formulation

The key concept behind the PoE formula is to regard each joint as applying a screw motion to all the outward links. To illustrate, consider a general spatial open chain like the one shown in Figure 5.8, consisting of n single dof joints that are connected serially. Choose a fixed frame, and also an end-effector frame attached to the last link—unlike the Denavit-Hartenberg convention, there are no restrictions placed on the choice of fixed and end-effector frames. Place the robot in its zero position by setting all joint values to zero, with the direction of positive displacement (rotation for revolute joints, translation for prismatic joints) for each joint specified.

Let $M \in SE(3)$ denote the configuration of the end-effector frame when the robot is in its zero position. Suppose only joint n is displaced to some joint value θ_n . The end-effector frame M then undergoes a screw motion of the form

$$T = e^{[\mathcal{S}_n]\theta_n} M, \quad (5.18)$$

where $T \in SE(3)$ is the newly displaced configuration of the end-effector frame, and $\mathcal{S}_n = (\omega_n, v_n)$ is the twist vector describing the motion for joint n . For example, if joint n is revolute (corresponding to a screw motion of zero pitch), then $\omega_n \in \mathbb{R}^3$ is a unit vector in the positive direction of joint axis n , and $v_n = -\omega_n \times q_n$, with q_n any arbitrary point on joint axis n . If joint n is prismatic, then $\omega_n = 0$, and $v \in \mathbb{R}^3$ is a unit vector in the direction of positive translation. We remind the reader that all vector quantities here are expressed in terms of the fixed frame.

If we assume joint $n - 1$ is also allowed to vary, then this has the effect of applying a screw motion to link $n - 1$ (and by extension to link n , since link n is connected to link $n - 1$ via joint n). The end-effector frame thus undergoes a screw motion of the form

$$T = e^{[\mathcal{S}_{n-1}]\theta_{n-1}} \left(e^{[\mathcal{S}_n]\theta_n} M \right). \quad (5.19)$$

Continuing with this reasoning and now allowing all the joints $(\theta_1, \dots, \theta_n)$ to vary, it follows that

$$T = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M. \quad (5.20)$$

This is the product of exponentials formula describing the forward kinematics of an n -dof open chain. Specifically, we will call Equation (5.20) the **space form** of the product of exponentials formula. We note the following features of the product of exponentials formula:

- No link reference frames need to be assigned. Only the fixed and end-effector frames need to be chosen (note that there are no restrictions on how these frames are chosen), together with a definition of the robot's zero position, and the direction of positive displacement for each joint (rotation in the case of a revolute joint, translation in the case of a prismatic joint).
- Placing the robot in its zero position, if joint i is revolute, then $\omega_i \in \mathbb{R}^3$ is a unit vector in the (positive) direction of the joint axis, and $v = -\omega_i \times q_i$, where $q_i \in \mathbb{R}^3$ is an arbitrary point on joint axis i . If joint i is prismatic, then $\omega_i = 0$ and $v_i \in \mathbb{R}^3$ is a unit vector in the direction of (positive) translation. Unlike the Denavit-Hartenberg parameters, where the joint variable involves either ϕ_i or d_i depending on whether the joint is revolute or prismatic, in the product of exponentials formula the joint variable is always labelled in a consistent fashion by θ_i .

Compared to the Denavit-Hartenberg parameters, the fact that no link reference frames are used, and that the fixed and end-effector frames can be assigned without any restrictions, considerably simplifies the forward kinematic analysis. Further advantages will come to light when we examine the velocity kinematics in the next chapter.

5.2.2 Examples

We now derive the forward kinematics for some common spatial open structures using the product of exponentials formula.

Example: 3R Spatial Open Chain

We return to the previous 3R open chain example of Figure 5.5. Choose the fixed frame $\{0\}$ and end-effector frame $\{3\}$ as indicated in the figure, and express all vectors and homogeneous transformations in terms of the fixed frame. The forward kinematics will be of the form

$$T = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M,$$

where $M \in SE(3)$ is the end-effector frame configuration when the robot is in its zero position. By inspection M can be obtained as

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The twist $\mathcal{S}_1 = (\omega_1, v_1)$ for joint axis 1 is then given by $\omega_1 = (0, 0, 1)$ and $v_1 = (0, 0, 0)$ (the fixed frame origin $(0,0,0)$ is a convenient choice for the point q_1 lying on joint axis 1). To determine the twist \mathcal{S}_2 for joint axis 2, observe that joint axis 2 points in the $-\hat{y}_0$ axis direction, so that $\omega_2 = (0, -1, 0)$. Choose $q_2 = (L_1, 0, 0)$, in which case $v_2 = -\omega_2 \times q_2 = (0, 0, -L_1)$. Finally, to determine the twist \mathcal{S}_3 for joint axis, note that $\omega_3 = (1, 0, 0)$. Choosing $q_3 = (0, 0, -L_2)$, it follows that $v_3 = -\omega_3 \times q_3 = (0, L_2, 0)$.

In summary, we have the following 4×4 matrix representations for the three joint twist vectors \mathcal{S}_1 , \mathcal{S}_2 , and \mathcal{S}_3 :

$$\begin{aligned} [\mathcal{S}_1] &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ [\mathcal{S}_2] &= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ [\mathcal{S}_3] &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

It will be more convenient to list the twist vectors in the following tabular form:

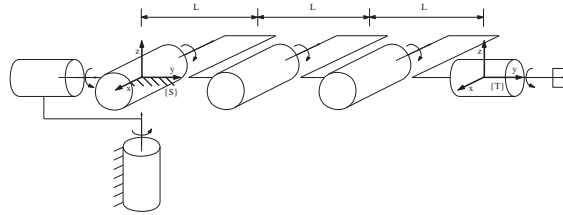


Figure 5.9: POE forward kinematics for the 6R open chain.

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, -1, 0)$	$(0, 0, -L_1)$
3	$(1, 0, 0)$	$(0, L_2, 0)$

Example: 6R Spatial Open Chain

We now derive the forward kinematics of the 6R open chain of Figure 5.9. The zero position and the direction of positive rotation for each joint axis are as shown in the figure. A fixed frame $\{0\}$ and end-effector frame $\{6\}$ are also assigned as shown. The end-effector frame M in the zero position is then

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.21)$$

The screw axis for joint 1 is in the direction $\omega_1 = (0, 0, 1)$. The most convenient choice for point q_1 lying on joint axis 1 is the origin, so that $v_1 = (0, 0, 0)$. The screw axis for joint 2 is in the \hat{y} direction of the fixed frame, so $\omega_2 = (0, 1, 0)$. Choosing $q_2 = (0, 0, 0)$, we have $v_2 = (0, 0, 0)$. The screw axis for joint 3 is in the direction $\omega_3 = (-1, 0, 0)$. Choosing $q_3 = (0, 0, 0)$ leads to $v_3(0, 0, 0)$. The screw axis for joint 4 is in the direction $\omega_4 = (-1, 0, 0)$. Choosing $q_4 = (0, L, 0)$ leads to $v_4 = (0, 0, -L)$. The screw axis for joint 5 is in the direction $\omega_5 = (-1, 0, 0)$; choosing $q_5 = (0, 2L, 0)$ leads to $v_5 = (0, 0, -2L)$. The screw axes for joint 6 is in the direction $\omega_6 = (0, 1, 0)$; choose $q_6 = (0, 0, 0)$ leads to $v_6 = (0, 0, 0)$. In summary, the twists $\mathcal{S}_i = (\omega_i, v_i)$, $i = 1, \dots, 6$ are as follows:

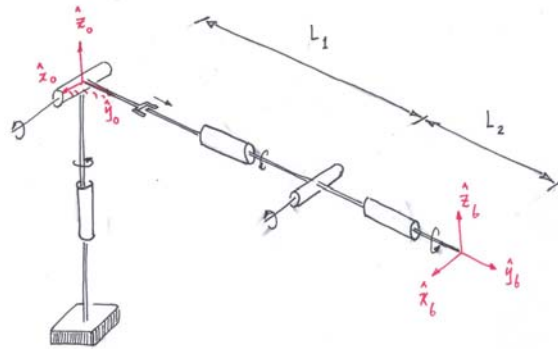


Figure 5.10: The RRPRRR spatial open chain.

i	ω_i	v_i
1	(0, 0, 1)	(0, 0, 0)
2	(0, 1, 0)	(0, 0, 0)
3	(-1, 0, 0)	(0, 0, 0)
4	(-1, 0, 0)	(0, 0, $-L$)
5	(-1, 0, 0)	(0, 0, $-2L$)
6	(0, 1, 0)	(0, 0, 0)

Example: A RRPRRR Spatial Open Chain

In this example we consider the six degree-of-freedom RRPRRR spatial open chain of Figure 5.10. The end-effector frame in the zero position is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The values of the screw twists $\mathcal{S}_i = (\omega_i, v_i)$ are listed in the following table:

i	ω_i	v_i
1	(0, 0, 1)	(0, 0, 0)
2	(1, 0, 0)	(0, 0, 0)
3	(0, 0, 0)	(0, 1, 0)
4	(0, 1, 0)	(0, 0, 0)
5	(1, 0, 0)	(0, 0, $-L_1$)
6	(0, 1, 0)	(0, 0, 0)

Note that the third joint is prismatic, so that $\omega_3 = 0$ and v_3 is a unit vector in the direction of positive translation.

5.2.3 Relation with the Denavit-Hartenberg Representation

The product of exponentials formula can also be derived directly from the Denavit-Hartenberg parameter-based representation of the forward kinematics. As before, denote the relative displacement between adjacent link frames by

$$T_{i-1,i} = \text{Rot}(\hat{x}, \alpha_{i-1}) \cdot \text{Trans}(\hat{x}, a_{i-1}) \cdot \text{Trans}(\hat{z}, d_i) \cdot \text{Rot}(\hat{z}, \phi_i).$$

If joint i is revolute, the first three matrices can be regarded as constant, and ϕ_i becomes the revolute joint variable. Define $\theta_i = \phi_i$, and

$$M_i = \text{Rot}(\hat{x}, \alpha_{i-1}) \cdot \text{Trans}(\hat{x}, a_{i-1}) \cdot \text{Trans}(\hat{z}, d_i), \quad (5.22)$$

and write $\text{Rot}(\hat{z}, \theta_i)$ as the following matrix exponential:

$$\text{Rot}(\hat{z}, \theta_i) = e^{[\mathcal{A}_i]\theta_i}, \quad [\mathcal{A}_i] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5.23)$$

With the above definitions we can write $T_{i-1,i} = M_i e^{[\mathcal{A}_i]\theta_i}$.

If joint i is prismatic, then d_i becomes the joint variable, ϕ_i is a constant parameter, and the order of $\text{Trans}(\hat{z}, d_i)$ and $\text{Rot}(\hat{z}, \phi_i)$ in $T_{i-1,i}$ can be reversed (recall that reversing translations and rotations taken along the same axis still results in the same motion). In this case we can still write $T_{i-1,i} = M_i e^{[\mathcal{A}_i]\theta_i}$, where $\theta_i = d_i$ and

$$M_i = \text{Rot}(\hat{x}, \alpha_{i-1}) \text{Trans}(\hat{x}, a_{i-1}) \text{Rot}(\hat{z}, \phi_i) \quad (5.24)$$

$$[\mathcal{A}_i] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5.25)$$

Based on the above, for an n -link open chain containing both revolute and prismatic joints, the forward kinematics can be written

$$T_{0,n} = M_1 e^{[\mathcal{A}_1]\theta_1} M_2 e^{[\mathcal{A}_2]\theta_2} \dots M_n e^{[\mathcal{A}_n]\theta_n} \quad (5.26)$$

where θ_i denotes joint variable i , and $[\mathcal{A}_i]$ is either of the form (5.23) or (5.25) depending on whether joint i is revolute or prismatic.

We now make use of the matrix identity $M e^P M^{-1} = e^{M P M^{-1}}$, which holds for any nonsingular $M \in \mathbb{R}^{n \times n}$ and arbitrary $P \in \mathbb{R}^{n \times n}$. The above can also be rearranged as $M e^P = e^{M P M^{-1}} M$. Beginning from the left of Equation (5.26), if we repeatedly apply this identity, after n iterations we obtain the product of exponentials formula as originally derived:

$$\begin{aligned} T_{0n} &= e^{M_1 [\mathcal{A}_1] M_1^{-1} \theta_1} (M_1 M_2) e^{[\mathcal{A}_2] \theta_2} \dots e^{[\mathcal{A}_n] \theta_n} \\ &= e^{M_1 [\mathcal{A}_1] M_1^{-1} \theta_1} e^{(M_1 M_2) [\mathcal{A}_2] (M_1 M_2)^{-1} \theta_2} (M_1 M_2 M_3) e^{[\mathcal{A}_3] \theta_3} \dots e^{[\mathcal{A}_n] \theta_n} \\ &= e^{[S_1] \theta_1} \dots e^{[S_n] \theta_n} M, \end{aligned} \quad (5.27)$$

where

$$[\mathcal{S}_i] = (M_1 \cdots M_{i-1})[\mathcal{A}_i](M_1 \cdots M_{i-1})^{-1}, \quad i = 1, \dots, n \quad (5.28)$$

$$M = M_1 M_2 \cdots M_n. \quad (5.29)$$

We now re-examine the physical meaning of the \mathcal{S}_i by recalling how a screw twist transforms under a change of reference frames. If \mathcal{S}_a represents the screw twist for a given screw motion with respect to frame $\{a\}$, and \mathcal{S}_b represents the screw twist for the same physical screw motion but this time with respect to frame $\{b\}$, then recall that \mathcal{S}_a and \mathcal{S}_b are related by

$$[\mathcal{S}_b] = T_{ba}[\mathcal{S}_a]T_{ba}^{-1}, \quad (5.30)$$

or using the adjoint notation $\text{Ad}_{T_{ba}}$,

$$\mathcal{S}_b = \text{Ad}_{T_{ba}}(\mathcal{S}_a). \quad (5.31)$$

Seen from the perspective of this transformation rule, Equation (5.29) suggests that \mathcal{A}_i is the screw twist for joint axis i as seen from link frame $\{i\}$, while \mathcal{S}_i is the screw twist for joint axis i as seen from the fixed frame $\{0\}$.

5.2.4 Second Formulation

The matrix identity $e^{M^{-1}PM} = M^{-1}e^P M$ can also be expressed as $M e^{M^{-1}PM} = e^P M$. Beginning with the right-most term of the previously derived product of exponentials formula, if we repeatedly apply this identity, after n iterations we obtain

$$\begin{aligned} T &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_n]\theta_n}, \end{aligned} \quad (5.32)$$

where each $[\mathcal{B}_i] = M^{-1}[\mathcal{S}_i]M$, $i = 1, \dots, n$. This is an alternative form of the product of exponentials formula. Note that M^{-1} is the relative displacement of the fixed frame as seen from the end-effector frame; since each

$$\mathcal{B}_i = \text{Ad}_{M^{-1}}(\mathcal{S}_i), \quad (5.33)$$

\mathcal{B}_i is the screw twist for joint i expressed in terms of the end-effector frame at the zero position. We will call Equation (5.32) the **body form** of the product of exponentials formula.

Example: 6R Spatial Open Chain

We now express the forward kinematics for the same 6R open chain of Figure 5.9 in the second form

$$T = M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_6]\theta_6}.$$

Assume the same fixed and end-effector frames and zero position as the previous example. M is still the same as Equation (5.21), obtained as the end-effector frame as seen from the fixed frame with the chain in its zero position. The screw twists for each joint axes, however, are now expressed with respect to the end-effector frame in its zero position:

i	ω_i	v_i
1	(0, 0, 1)	(-3L, 0, 0)
2	(0, 1, 0)	(0, 0, 0)
3	(-1, 0, 0)	(0, 0, -3L)
4	(-1, 0, 0)	(0, 0, -2L)
5	(-1, 0, 0)	(0, 0, -L)
6	(0, 1, 0)	(0, 0, 0)

5.3 Summary

- Given an open chain with a reference frame attached to some point on its last link—this frame is denoted the **end-effector frame**—the forward kinematics is the mapping from the joint values to the position and orientation of the end-effector frame.
- In the **Denavit-Hartenberg** representation, the forward kinematics of an open chain is described in terms of relative displacements between reference frames attached to each link. If the link frames are sequentially labelled $\{0\}, \dots, \{n\}$, where $\{0\}$ is the fixed frame and $\{n\}$ is the end-effector frame, then the forward kinematics is expressed as

$$T_{0n} = T_{01}(\theta_1) \cdots T_{n-1,n}(\theta_n)$$

where θ_i denotes the joint i variable.

- The Denavit-Hartenberg convention requires that reference frames assigned to each link obey a strict convention. Following this convention, the link frame transformation $T_{i-1,i}$ between link frames $\{i-1\}$ and $\{i\}$ can be parametrized using a minimum of four parameters (the **Denavit-Hartenberg parameters**): the link twist α_{i-1} , link length a_{i-1} , link offset d_i , and joint angle ϕ_i .
- The forward kinematics can also be expressed as the following **product of exponentials** (the **space form**):

$$T_{0n} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M,$$

where $\mathcal{S}_i = (\omega_i, v_i)$ denotes the screw twist associated with joint i expressed in fixed frame coordinates, θ_i is the joint i variable, and $M \in SE(3)$ denotes the position and orientation of the end-effector frame when the robot is in its zero position. A choice of fixed frame and end-effector

frame, together with a specification of the robot's zero position and direction of positive rotation or translation of each of the robot's joints, then completely specifies the product of exponentials formula.

- The product of exponentials formula can also be written in the equivalent **body form**:

$$T_{0n} = M e^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_n]\theta_n},$$

where $[\mathcal{B}]_i = M^{-1}[\mathcal{S}_i]M$, $i = 1, \dots, n$. $\mathcal{B}_i = (\omega_i, v_i)$ is the screw twist corresponding to joint axis i , expressed in terms of the end-effector frame with the robot in its zero position.

Notes and References

In the original paper by Denavit and Hartenberg [7], the four parameters are defined without appealing to any reference frames, e.g., the link length is defined to be length of the mutually perpendicular line between the joint axes, the twist angle is obtained by first projecting the two adjacent axes to the plane perpendicular to the common normal and measuring the angle between the axes, etc. The convention of choosing the \hat{z} -axis to be along the joint axis and the \hat{x} -axis to be along the common normal is not standard; other authors, for example, choose the joint axis to be in the \hat{x} -direction. The product of exponentials formula is first presented by Brockett in [4]. Computational aspects of the product of exponentials formula are also discussed in [27].