Chapter 4

Rigid-Body Motions

In the previous chapter, we saw that a minimum of six numbers are needed to specify the position and orientation of a rigid body in three-dimensional physical space. We established this by selecting three points on a rigid body, and arguing that the distances between any pair of these three points must always be preserved regardless of where the rigid body is located. This led to three constraints, which when imposed on the nine Cartesian coordinates—(x, y, z) coordinates for each of the three points—led us to conclude that only six of these nine coordinates could be independently chosen.

In this chapter we will develop a more systematic way to describe the position and orientation of a rigid body. Rather than choosing three points on a body, we instead attach a reference frame to the body, and develop ways to describe this reference frame with respect to some fixed reference frame in space (we know of course that this can be done using as few as six coordinates). This is the **descriptive** aspect of the rigid body motions.

There is also a **prescriptive** aspect to the rigid body motions. Suppose a rigid body is moved from one configuration in physical space to another. Once a reference frame and length scale for physical space have been chosen, the displacement of this rigid body can then described by a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. It turns out that the same set of mathematical representations can be used for both the descriptive and prescriptive interpretations of rigid body motions.

To illustrate the simultaneous descriptive-prescriptive features of rigid body motions, and also to provide a synopsis of the main concepts and tools that we will learn in this chapter, we begin with a motivating planar example. Before doing so, we make some remarks about vector notation.

A Word about Vector Notation

Recall that a vector is a geometric quantity with a length and direction. A vector will be denoted by a regular text symbol, e.g., v. If a reference frame and length scale have been chosen for the underlying space in which the vector

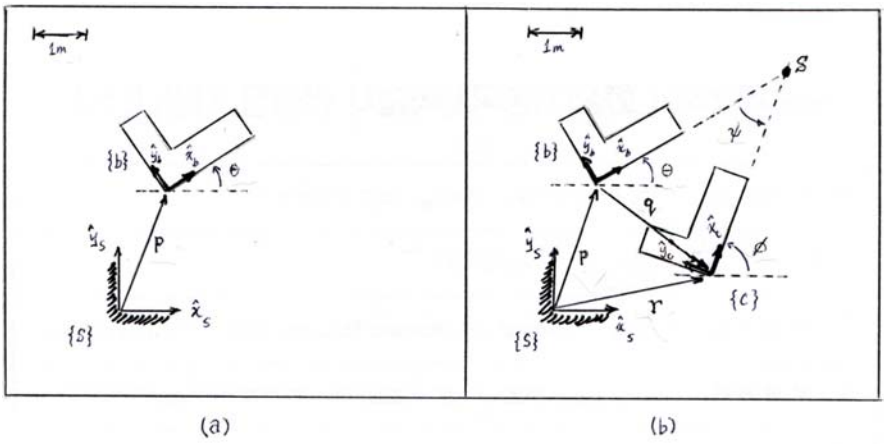

Figure 4.1: Rigid body motion in the plane.

v lies, then this vector can be represented as a column vector with respect to this reference frame (imagine moving the base of v to the reference frame origin while maintaining its direction; such a notion of vector is sometimes referred to as a **free vector**). The column vector representation of v will be denoted in italics by $v \in \mathbb{R}^n$. Note that if a different reference frame and length scale are chosen, then the column vector representation v will change.

A point p in physical space can also be represented as a vector. Given a choice of reference frame and length scale for physical space, the point p can be represented as a vector from the reference frame origin to p; its column vector representation will be denoted in italics by $p \in \mathbb{R}^n$. Here as before, a different choice of reference frame and length scale for physical space will lead to a different column vector representation $p \in \mathbb{R}^n$ for the same point p in physical space.

4.1 A Motivating Example

Consider the planar body of Figure 4.1(a), whose motion is confined to the plane. Suppose a length scale and a fixed reference frame have been chosen as shown. We will call the fixed reference frame the **fixed frame**, or the **space frame**, denoted {s}, and label its unit axes $\hat{\mathrm{x}}_\mathrm{s}$ and $\hat{\mathrm{y}}_\mathrm{s}$. Similarly, we attach a reference frame with unit axes $\hat{\mathrm{x}}_\mathrm{b}$ and $\hat{\mathrm{y}}_\mathrm{b}$ to the planar body. Because this frame moves with the body, it will be called the **moving frame**, or **body frame**, and denoted {b}.

To describe the configuration of the planar body, only the position and orientation of the body frame with respect to the fixed frame needs to be specified. The vector from the fixed frame origin to the body frame origin, denoted p, can be expressed in terms of the fixed frame unit axes as

$$\mathrm{p} = p_x \hat{\mathrm{x}}_\mathrm{s} + p_y \hat{\mathrm{y}}_\mathrm{s}. \tag{4.1}$$

You are probably more accustomed to writing this vector as simply (p_x, p_y); this is fine when there is no possibility of ambiguity about reference frames, but when expressing the same vector in terms of multiple reference frames, writing p as in Equation (4.1) clearly indicates the reference frame with respect to which (p_x, p_y) is defined.

The simplest way to describe the orientation of the body frame {b} relative to the fixed frame {s} is by specifying the angle θ as shown in the figure. Another (admittedly less simple) way is to specify the directions of the unit axes $\hat{\mathrm{x}}_\mathrm{b}$ and $\hat{\mathrm{y}}_\mathrm{b}$ of the body frame, in the form

$$\hat{\mathrm{x}}_\mathrm{b} = \cos\theta\, \hat{\mathrm{x}}_\mathrm{s} + \sin\theta\, \hat{\mathrm{y}}_\mathrm{s}, \tag{4.2}$$

$$\hat{\mathrm{y}}_\mathrm{b} = -\sin\theta\, \hat{\mathrm{x}}_\mathrm{s} + \cos\theta\, \hat{\mathrm{y}}_\mathrm{s}. \tag{4.3}$$

At first sight this seems a rather inefficient way to represent the body frame orientation. However, imagine if now the body were to move arbitrarily in three-dimensional space; a single angle θ alone clearly would not suffice to describe the orientation of the displaced reference frame. We would in fact need three angles, but as yet it is not clear how to define an appropriate set of three angles. On the other hand, expressing the displaced body frame's unit axes in terms of the fixed frame, as we have done above for the planar case, is straightforward.

Assuming we agree to express everything in terms of the fixed frame coordinates, then just as the vector p of Equation (4.1) can be represented as a column vector $p \in \mathbb{R}^2$ of the form

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}, \tag{4.4}$$

Equations (4.2)-(4.3) can also be packaged into the following 2×2 matrix P:

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{4.5}$$

Observe that the first column of P corresponds to $\hat{\mathrm{x}}_\mathrm{b}$, and the second column to $\hat{\mathrm{y}}_\mathrm{b}$. It can be easily verified that that $P^T P = I$, and that $P^{-1} = P^T$. The matrix P as constructed here is an example of a **rotation matrix**, and the pair (P, p) provides a description of the orientation and position of the body frame relative to the fixed frame. Of the six entries involved—two for p and four for P—only three are independent. In fact, the condition $P^T P = I$ implies three equality constraints, so that of the four entries that make up P, only one can be chosen independently. The three coordinates (θ, p_x, p_y) would seem to offer an intuitive minimal representation for the configuration of a planar rigid body. In any event, the rotation matrix-vector pair (P, p) serves as a description of the configuration of the rigid body as seen from the fixed frame.

Now referring to Figure 4.1(b), suppose the body at {b} is displaced to the configuration at {c}. Repeating the above analysis for the position vector r and the unit axes of frame {c}, we can write

$$
\begin{aligned}
\mathrm{r} &= r_x \hat{\mathrm{x}}_\mathrm{s} + r_y \hat{\mathrm{y}}_\mathrm{s} && (4.6)\\
\hat{\mathrm{x}}_\mathrm{c} &= \cos\phi\,\hat{\mathrm{x}}_\mathrm{s} + \sin\phi\,\hat{\mathrm{y}}_\mathrm{s}, && (4.7)\\
\hat{\mathrm{y}}_\mathrm{c} &= -\sin\phi\,\hat{\mathrm{x}}_\mathrm{s} + \cos\phi\,\hat{\mathrm{y}}_\mathrm{s}. && (4.8)
\end{aligned}
$$

The above can be arranged into the following column vector $r \in \mathbb{R}^2$ and rotation matrix $R \in \mathbb{R}^{2\times 2}$:

$$
r = \begin{bmatrix} r_x \\ r_y \end{bmatrix}, \quad R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{4.9}
$$

We could also repeat the above to describe frame {c} as seen from frame {b} (that is, pretend {b} is now the fixed frame, and repeat the previous analysis for {c}). Letting q denote the vector from the {b}-frame origin to the {c}-frame origin, we get

$$
\begin{aligned}
\mathrm{q} &= q_x \hat{\mathrm{x}}_\mathrm{b} + q_y \hat{\mathrm{y}}_\mathrm{b}, && (4.10)\\
\hat{\mathrm{x}}_\mathrm{c} &= \cos\psi\,\hat{\mathrm{x}}_\mathrm{b} + \sin\psi\,\hat{\mathrm{y}}_\mathrm{b}, && (4.11)\\
\hat{\mathrm{y}}_\mathrm{c} &= -\sin\psi\,\hat{\mathrm{x}}_\mathrm{b} + \cos\psi\,\hat{\mathrm{y}}_\mathrm{b}, && (4.12)
\end{aligned}
$$

where $\psi = \phi - \theta$. As before, the vector $q \in \mathbb{R}^2$ and rotation matrix $Q \in \mathbb{R}^{2\times 2}$ can be defined as follows:

$$
q = \begin{bmatrix} q_x \\ q_y \end{bmatrix}, \quad Q = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}. \tag{4.13}
$$

Here q is the representation of q in frame {b} coordinates. Similarly, the rotation matrix Q describes the orientation of frame {c} relative to frame {b}.

Now, if we imagine the body is displaced from frame {b} to frame {c}, then clearly the point on the rigid body corresponding to p is displaced to r. We can then ask, how is the column vector representation p transformed into r? The answer is given by

$$
r = p + Pq. \tag{4.14}
$$

To see why, from Figure 4.1(b) it can be seen that r is the vector sum of p and q. Since p and r are respectively the column vector representations of p and r in frame $\{s\}$ coordinates, to obtain r as a function of p and q we first need to represent q in frame $\{s\}$ coordinates before adding this to p. This is precisely Pq (verify this for yourself), and Equation (4.14) now follows accordingly. Since $\theta + \psi = \phi$, it can also be verified that

$$
R = PQ. \tag{4.15}
$$

The pair (Q, q) thus describes how the rigid body is displaced from {b} to {c}: given any point p on the rigid body at configuration {b}, represented by the vector $p \in \mathbb{R}^3$, it is then transformed to the point in physical space corresponding to the vector $r = p + Pq \in \mathbb{R}^3$. Also, the rotation matrix P

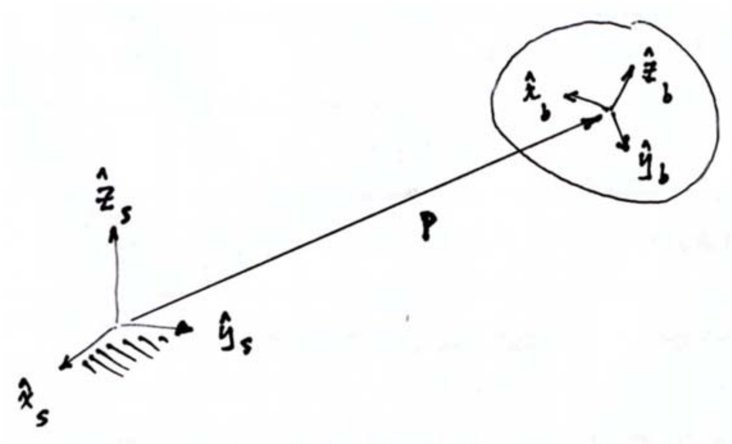

Figure 4.2: Mathematical description of position and orientation.

at configuration {b} is now transformed to $R = PQ$ at configuration {c} as a result of this displacement. For this reason we call the rotation matrix-vector pair (Q, q) a **rigid body displacement**, or more commonly a **rigid body motion**.

We thus see that a rotation matrix-vector pair can serve as a description of a rigid body's configuration (the descriptive interpretation as illustrated by (P, p)), or as a prescription of a rigid body's displacement in physical space (the prescriptive interpretation as illustrated by (Q, q)).

We make one final observation. Once again referring to Figure 4.1(b), note that the displacement from configuration {b} to {c} can be obtained by rotating the planar body at {b} about the point s by an angle ψ. The displacement can therefore be parametrized by the three coordinates (ψ, s_x, s_y), where (s_x, s_y) denote the coordinates for point s in fixed frame coordinates. This alternative three-parameter representation of a rigid body motion is a (planar) example of a **screw motion**. If the displacement is a pure translation—that is, the orientation does not change, so that $\theta = \phi$—then s is assumed to lie at infinity.

In the remainder of this chapter we will generalize the above concepts to three-dimensional rigid body motions. For this purpose consider a rigid body occupying three-dimensional physical space as shown in Figure 4.2. Assume that a length scale for physical space has been chosen, and that both the fixed frame {s} and body frame {b} have been chosen as shown. Throughout this book all reference frames will be right-handed, i.e., the unit axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$ always satisfy $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$. Denote the unit axes of the fixed frame by $\{\hat{\mathrm{x}}_\mathrm{s}, \hat{\mathrm{y}}_\mathrm{s}, \hat{\mathrm{z}}_\mathrm{s}\}$, and the unit axes of the body frame by $\{\hat{\mathrm{x}}_\mathrm{b}, \hat{\mathrm{y}}_\mathrm{b}, \hat{\mathrm{z}}_\mathrm{b}\}$. Let p denote the vector from the fixed frame origin to the body frame origin. In terms of the fixed frame coordinates, p can be expressed as

$$\mathrm{p} = p_1\hat{\mathrm{x}}_\mathrm{s} + p_2\hat{\mathrm{y}}_\mathrm{s} + p_3\hat{\mathrm{z}}_\mathrm{s} \tag{4.16}$$

The axes of the body frame can also be expressed as

$$\begin{aligned} \hat{\mathrm{x}}_{\mathrm{b}} &= r_{11}\hat{\mathrm{x}}_{\mathrm{s}} + r_{21}\hat{\mathrm{y}}_{\mathrm{s}} + r_{31}\hat{\mathrm{z}}_{\mathrm{s}} && (4.17)\\ \hat{\mathrm{y}}_{\mathrm{b}} &= r_{12}\hat{\mathrm{x}}_{\mathrm{s}} + r_{22}\hat{\mathrm{y}}_{\mathrm{s}} + r_{32}\hat{\mathrm{z}}_{\mathrm{s}} && (4.18)\\ \hat{\mathrm{z}}_{\mathrm{b}} &= r_{13}\hat{\mathrm{x}}_{\mathrm{s}} + r_{23}\hat{\mathrm{y}}_{\mathrm{s}} + r_{33}\hat{\mathrm{z}}_{\mathrm{s}}. && (4.19) \end{aligned}$$

Defining $p \in \mathbb{R}^3$ and $R \in \mathbb{R}^{3\times 3}$ by

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \tag{4.20}$$

the twelve parameters given by (R, p) then provide a description of the position and orientation of the rigid body relative to the fixed frame.

Since a minimum of six parameters are required to describe the configuration of a rigid body, if we agree to keep the three parameters in p as they are, then of the nine parameters in R, only three can be chosen independently. We begin by examining some basic three-parameter representations for rotation matrices: the **Euler angles** and the related **roll-pitch-yaw angles**, the **exponential coordinates**, and the **unit quaternions**. We then examine six-parameter representations for the combined position and orientation of a rigid body. Augmenting the three-parameter representation for R with $p \in \mathbb{R}^3$ is one obvious and natural way to do this. Another relies on the Chasles-Mozzi Theorem, which states that every rigid body displacement can be described as a screw motion about some fixed axis in space.

We conclude with a discussion of linear and angular velocities, and also of forces and moments. Rather than treat these quantities as separate three-dimensional quantities, we shall merge the linear and angular velocity vectors into a single six-dimensional **spatial velocity**, and also the moment and force vectors into a six-dimensional **spatial force**. These six-dimensional quantities, and the rules for manipulating them, will form the basis for the kinematic and dynamic analyses in the subsequent chapters.

4.2 Rotations

4.2.1 Definition

We argued earlier that of the nine entries in the rotation matrix R, only three can be chosen independently. We begin by deriving a set of six explicit constraints on the entries of R. Note that the three columns of R correspond to the body frame's unit axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$. The following conditions must therefore be satisfied:

(i) Unit norm condition: $\hat{\mathrm{x}}$, $\hat{\mathrm{y}}$, and $\hat{\mathrm{z}}$ are all of unit norm, or

$$\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\ r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \\ r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1. \end{aligned} \tag{4.21}$$

(ii) Orthogonality condition: $\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$ (here $\cdot$ denotes the inner product), or

$$\begin{aligned} r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0 \\ r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} &= 0 \\ r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} &= 0. \end{aligned} \tag{4.22}$$

These six constraints can be expressed more compactly as a single constraint on the matrix R:

$$R^T R = I, \tag{4.23}$$

where R^T denotes the transpose of R, and I denotes the 3×3 identity.

There is still the small matter of accounting for the fact that the frame is right-handed (i.e., $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, where $\times$ denotes the cross-product) rather than left-handed (i.e., $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{z}}$); our six equality constraints above do not distinguish between right- and left-handed frames. We recall the following formula for evaluating the determinant of a 3×3 matrix M: denoting its three columns by a, b, c, respectively, its determinant is given by

$$\det M = a^T(b \times c) = c^T(a \times b) = b^T(c \times a). \tag{4.24}$$

Substituting the columns for R into this formula then leads to the constraint

$$\det R = 1. \tag{4.25}$$

Note that if the frame had been left-handed, we would have $\det R = -1$. In summary, the six equality constraints represented by (4.23) imply that $\det R = \pm 1$; imposing the additional constraint $\det R = 1$ means that only right-handed frames are allowed. Thus, the constraint $\det R = 1$ does not change the number of independent continuous variables needed to parametrize R.

The set of 3×3 rotation matrices forms the **Special Orthogonal Group** $SO(3)$, which we now formally define:

Definition 4.1. The **Special Orthogonal Group** $SO(3)$, also known as the group[1] of rotation matrices, is the set of all 3×3 real matrices R that satisfy (i) $R^T R = I$, and (ii) $\det R = 1$.

The set of 2×2 rotation matrices is a subgroup of $SO(3)$, and denoted $SO(2)$.

Definition 4.2. The **Special Orthogonal Group** $SO(2)$ is the set of all 2×2 real matrices R that satisfy (i) $R^T R = I$, and (ii) $\det R = 1$.

From the definition it follows that every $R \in SO(2)$ is of the form

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where $\theta \in [0, 2\pi]$. In what follows, all properties derived for $SO(3)$ also apply to $SO(2)$.

[1] The formal algebraic notion of a group is discussed in the exercises.

4.2.2 Properties

We now list some basic properties of rotation matrices. First, the identity I is a trivial example of a rotation matrix. The inverse of a rotation matrix is also a rotation matrix:

Proposition 4.1. *The inverse of a rotation matrix $R \in SO(3)$ always exists and is given by its transpose, R^T. R^T is also a rotation matrix.*

Proof. The condition $R^T R = I$ implies that $R^{-1} = R^T$, which clearly exists for every R. Since $\det R^T = \det R = 1$, R^T is also a rotation matrix. Another consequence of the equality $R^T = R^{-1}$ is that $RR^T = I$. □

Proposition 4.2. *The product of two rotation matrices is a rotation matrix.*

Proof. Given $R_1, R_2 \in SO(3)$, it readily follows that their product $R_1 R_2$ satisfies $(R_1 R_2)^T (R_1 R_2) = I$ and $\det R_1 R_2 = \det R_1 \cdot \det R_2 = 1$. □

Proposition 4.3. *For any vector $x \in \mathbb{R}^3$ and $R \in SO(3)$, the vector $y = Rx$ is of the same length as x.*

Proof. This follows from $\|y\|^2 = y^T y = x^T R^T R x = x^T x = \|x\|^2$. □

The next property provides a descriptive interpretation for the product of two rotation matrices. We first introduce some notation. Given two reference frames {a} and {b}, the orientation of frame {b} as seen from frame {a} will be represented by the rotation matrix R_{ab}; that is, the three columns of R_{ab} are just vector representations of the $\hat{\mathrm{x}}$-, $\hat{\mathrm{y}}$-, and $\hat{\mathrm{z}}$-axes of frame {b} expressed in terms of coordinates for frame {a}. From this definition it follows readily that $R_{aa} = I$.

Proposition 4.4. $R_{ab} R_{bc} = R_{ac}$.

Proof. To prove this result, introduce a third reference frame {c}, and define the unit axes of frames {a}, {b}, and {c} by the triplet of orthogonal unit vectors $\{\hat{\mathrm{x}}_{\mathrm{a}}, \hat{\mathrm{y}}_{\mathrm{a}}, \hat{\mathrm{z}}_{\mathrm{a}}\}$, $\{\hat{\mathrm{x}}_{\mathrm{b}}, \hat{\mathrm{y}}_{\mathrm{b}}, \hat{\mathrm{z}}_{\mathrm{b}}\}$, and $\{\hat{\mathrm{x}}_{\mathrm{c}}, \hat{\mathrm{y}}_{\mathrm{c}}, \hat{\mathrm{z}}_{\mathrm{c}}\}$, respectively. Suppose that the unit axes of frame {b} can be expressed in terms of the unit axes of frame {a} by

$$\begin{aligned}
\hat{\mathrm{x}}_{\mathrm{b}} &= r_{11}\hat{\mathrm{x}}_{\mathrm{a}} + r_{21}\hat{\mathrm{y}}_{\mathrm{a}} + r_{31}\hat{\mathrm{z}}_{\mathrm{a}} \\
\hat{\mathrm{y}}_{\mathrm{b}} &= r_{12}\hat{\mathrm{x}}_{\mathrm{a}} + r_{22}\hat{\mathrm{y}}_{\mathrm{a}} + r_{32}\hat{\mathrm{z}}_{\mathrm{a}} \\
\hat{\mathrm{z}}_{\mathrm{b}} &= r_{13}\hat{\mathrm{x}}_{\mathrm{a}} + r_{23}\hat{\mathrm{y}}_{\mathrm{a}} + r_{33}\hat{\mathrm{z}}_{\mathrm{a}}.
\end{aligned}$$

Similarly, suppose the unit axes of frame $\{c\}$ can be expressed in terms of the unit axes of frame $\{b\}$ by

$$\begin{aligned}
\hat{\mathrm{x}}_{\mathrm{c}} &= s_{11}\hat{\mathrm{x}}_{\mathrm{b}} + s_{21}\hat{\mathrm{y}}_{\mathrm{b}} + s_{31}\hat{\mathrm{z}}_{\mathrm{b}} \\
\hat{\mathrm{y}}_{\mathrm{c}} &= s_{12}\hat{\mathrm{x}}_{\mathrm{b}} + s_{22}\hat{\mathrm{y}}_{\mathrm{b}} + s_{32}\hat{\mathrm{z}}_{\mathrm{b}} \\
\hat{\mathrm{z}}_{\mathrm{c}} &= s_{13}\hat{\mathrm{x}}_{\mathrm{b}} + s_{23}\hat{\mathrm{y}}_{\mathrm{b}} + s_{33}\hat{\mathrm{z}}_{\mathrm{b}}.
\end{aligned}$$

The above equations can also be expressed more compactly as

$$\begin{bmatrix} \hat{\mathrm{x}}_\mathrm{b} & \hat{\mathrm{y}}_\mathrm{b} & \hat{\mathrm{z}}_\mathrm{b} \end{bmatrix} = \begin{bmatrix} \hat{\mathrm{x}}_\mathrm{a} & \hat{\mathrm{y}}_\mathrm{a} & \hat{\mathrm{z}}_\mathrm{a} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (4.26)$$

$$\begin{bmatrix} \hat{\mathrm{x}}_\mathrm{c} & \hat{\mathrm{y}}_\mathrm{c} & \hat{\mathrm{z}}_\mathrm{c} \end{bmatrix} = \begin{bmatrix} \hat{\mathrm{x}}_\mathrm{b} & \hat{\mathrm{y}}_\mathrm{b} & \hat{\mathrm{z}}_\mathrm{b} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}. \quad (4.27)$$

Note that the 3×3 matrix of Equation (4.26) is R_{ab}, while that of Equation (4.27) is R_{bc}. Substituting (4.26) into (4.27),

$$\begin{bmatrix} \hat{\mathrm{x}}_\mathrm{c} & \hat{\mathrm{y}}_\mathrm{c} & \hat{\mathrm{z}}_\mathrm{c} \end{bmatrix} = \begin{bmatrix} \hat{\mathrm{x}}_\mathrm{a} & \hat{\mathrm{y}}_\mathrm{a} & \hat{\mathrm{z}}_\mathrm{a} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}.$$

From the above it follows that $R_{ab}R_{bc} = R_{ac}$. □

Proposition 4.5.

$$R_{ab}^{-1} = R_{ba}. \quad (4.28)$$

Proof. This result follows immediately by choosing {c} to be the same as {a} in the previous proposition $R_{ab}R_{bc} = R_{ac}$, and recalling that $R_{aa} = I$. □

For the next property, consider a free vector v in physical space with a defined direction and magnitude. Given two reference frames {a} and {b} in physical space, let $v_a, v_b \in \mathbb{R}^3$ denote representations of v with respect to these two frames; that is, v_a and v_b are obtained by placing the base of v at the origin of frames {a} and {b}, respectively, and expressing v in terms of the given reference frame.

Proposition 4.6.

$$R_{ba}v_a = v_b \quad (4.29)$$

$$R_{ab}v_b = v_a. \quad (4.30)$$

Proof. In terms of the unit axes of {a} and {b}, v can be written as follows:

$$\mathrm{v} = v_{a_1}\hat{\mathrm{x}}_\mathrm{a} + v_{a_2}\hat{\mathrm{y}}_\mathrm{a} + v_{a_3}\hat{\mathrm{z}}_\mathrm{a} \quad (4.31)$$

$$= v_{b_1}\hat{\mathrm{x}}_\mathrm{b} + v_{b_2}\hat{\mathrm{y}}_\mathrm{b} + v_{b_3}\hat{\mathrm{z}}_\mathrm{b}, \quad (4.32)$$

or, letting $v_a = (v_{a_1}, v_{a_2}, v_{a_3})^T$, $v_b = (v_{b_1}, v_{b_2}, v_{b_3})^T$,

$$\mathbf{v} = \begin{bmatrix} \hat{\mathrm{x}}_\mathrm{a} & \hat{\mathrm{y}}_\mathrm{a} & \hat{\mathrm{z}}_\mathrm{a} \end{bmatrix} v_a \quad (4.33)$$

$$= \begin{bmatrix} \hat{\mathrm{x}}_\mathrm{b} & \hat{\mathrm{y}}_\mathrm{b} & \hat{\mathrm{z}}_\mathrm{b} \end{bmatrix} v_b. \quad (4.34)$$

From the previous proposition the unit axes of {a} and {b} are related by

$$\begin{bmatrix} \hat{\mathrm{x}}_\mathrm{a} & \hat{\mathrm{y}}_\mathrm{a} & \hat{\mathrm{z}}_\mathrm{a} \end{bmatrix} = \begin{bmatrix} \hat{\mathrm{x}}_\mathrm{b} & \hat{\mathrm{y}}_\mathrm{b} & \hat{\mathrm{z}}_\mathrm{b} \end{bmatrix} R_{ba}, \quad (4.35)$$

from which it follows that $R_{ba}v_a = v_b$. □

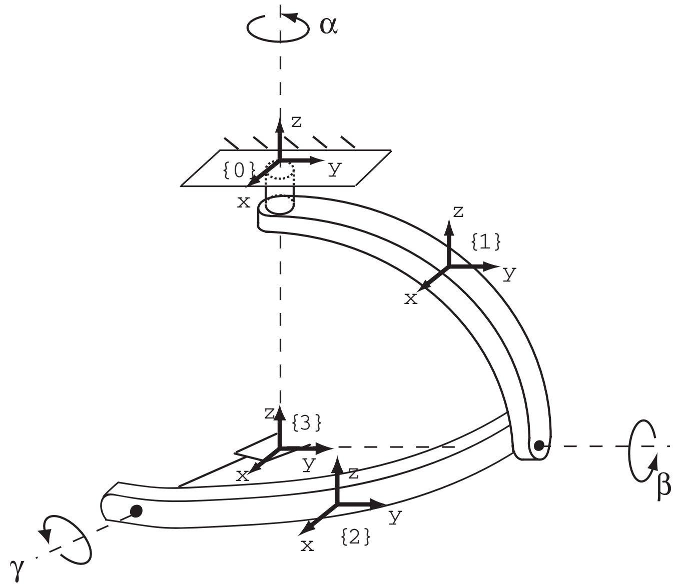

Figure 4.3: Wrist mechanism illustrating the ZYX Euler angles.

4.2.3 Euler Angles

As we established above, a rotation matrix can be parametrized by three independent coordinates. Here we introduce one popular three-parameter representation of rotations, the **ZYX Euler angles**. One way to visualize these angles is through the wrist mechanism shown in Figure 4.3. The ZYX Euler angles (α, β, γ) refer to the angle of rotation about the three joint axes of this mechanism. In the figure the wrist mechanism is shown in its zero position, i.e., when all three joints are set to zero.

Four reference frames are defined as follows: frame $\{0\}$ is the fixed frame, while frames $\{1\}$, $\{2\}$, and $\{3\}$ are attached to the three links of the wrist mechanism as shown. When the wrist is in the zero position, all four reference frames have the same orientation. We now consider the relative frame orientations R_{01}, R_{12}, and R_{23}. First, it can be seen that R_{01} depends only on the angle α: rotating about the $\hat{z}$-axis of frame $\{0\}$ by an angle α (a positive rotation about an axis is taken by aligning the thumb of the right hand along the axis, and rotating in the direction of the fingers curling about the axis), it can be seen that

$$R_{01} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{Rot}(\hat{z}, \alpha). \tag{4.36}$$

The notation $\mathrm{Rot}(\hat{z}, \alpha)$ describes a rotation about the $\hat{z}$-axis by angle α. Similarly, R_{12} depends only on β, and is given by

$$R_{12} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} = \mathrm{Rot}(\hat{\mathrm{y}}, \beta), \tag{4.37}$$

where the notation $\mathrm{Rot}(\hat{\mathrm{y}}, \beta)$ describes a rotation about the $\hat{\mathrm{y}}$-axis by angle β. Finally, R_{23} depends only on γ, and is given by

$$R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} = \mathrm{Rot}(\hat{\mathrm{x}}, \gamma), \tag{4.38}$$

where the notation $\mathrm{Rot}(\hat{\mathrm{x}}, \gamma)$ describes a rotation about the $\hat{\mathrm{x}}$-axis by angle γ. $R_{03} = R_{01}R_{12}R_{23}$ is therefore given by

$$R_{03} = \begin{bmatrix} \mathrm{c}_\alpha \mathrm{c}_\beta & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma - \mathrm{s}_\alpha \mathrm{c}_\gamma & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma + \mathrm{s}_\alpha \mathrm{s}_\gamma \\ \mathrm{s}_\alpha \mathrm{c}_\beta & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma + \mathrm{c}_\alpha \mathrm{c}_\gamma & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma - \mathrm{c}_\alpha \mathrm{s}_\gamma \\ -\mathrm{s}_\beta & \mathrm{c}_\beta \mathrm{s}_\gamma & \mathrm{c}_\beta \mathrm{c}_\gamma \end{bmatrix}, \tag{4.39}$$

where s_α is shorthand for $\sin\alpha$, c_α for $\cos\alpha$, etc.

We now ask the following question: given an arbitrary rotation matrix R, does there exist (α, β, γ) such that Equation (4.39) is satisfied? If the answer is yes, then the wrist mechanism of Figure 4.3 can reach any orientation. This is indeed the case, and we prove this fact constructively as follows. Let r_{ij} be the ij-th element of R. Then from Equation (4.39) we know that $r_{11}^2 + r_{21}^2 = \cos^2\beta$; as long as $\cos\beta \neq 0$, or equivalently $\beta \neq \pm 90°$, we have

$$\beta = \tan^{-1}\left(\frac{\sin\beta}{\cos\beta}\right) = \tan^{-1}\left(\frac{-r_{31}}{\pm\sqrt{r_{11}^2 + r_{21}^2}}\right).$$

We now define the atan2 function, which is a two-argument function implemented in a variety of computer languages for computing the arctangent. Specifically, the function $\mathrm{atan2}(y, x)$ evaluates $\tan^{-1}(y/x)$ by taking into account the signs of x and y. For example, $\mathrm{atan2}(1, 1) = \pi/4$, while $\mathrm{atan2}(-1, -1) = -3\pi/4$. Using atan2, the possible values for β can be expressed as

$$\beta = \mathrm{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

and

$$\beta = \mathrm{atan2}\left(-r_{31}, -\sqrt{r_{11}^2 + r_{21}^2}\right).$$

In the first case β will lie in the range $[-90°, 90°]$, while in the second case β lies in the range $[90°, 270°]$. Assuming the β obtained above is not $\pm 90°$, α and γ can then be determined from the following relations:

$$\begin{aligned} \alpha &= \mathrm{atan2}(r_{21}, r_{11}) \\ \gamma &= \mathrm{atan2}(r_{32}, r_{33}) \end{aligned}$$

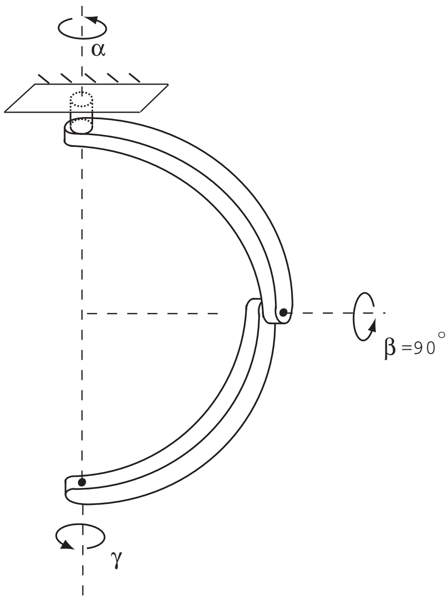

Figure 4.4: Configuration corresponding to $\beta = 90°$ for ZYX Euler angles.

In the event that $\beta = \pm 90°$, there exists a one-parameter family of solutions for α and γ. This is most easily seen from Figure 4.4. If $\beta = 90°$, then α and γ represent rotations (in the opposite direction) about the same vertical axis. Hence, if $(\alpha, \beta, \gamma) = (\bar{\alpha}, 90°, \bar{\gamma})$ is a solution for a given rotation R, then any triple $(\bar{\alpha}', 90°, \bar{\gamma}')$ where $\bar{\alpha}' - \bar{\gamma}' = \bar{\alpha} - \bar{\gamma}$ is also a solution.

Algorithm for Computing the ZYX Euler Angles

Given $R \in SO(3)$, we wish to find angles $\alpha, \gamma \in [0, 2\pi)$ and $\beta \in [-\pi/2, \pi/2)$ that satisfy

$$R = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}, \tag{4.40}$$

where s_α is shorthand for $\sin\alpha$, c_α for $\cos\alpha$, etc. Denote by r_{ij} the ij-th entry of R.

(i) If $r_{31} \neq \pm 1$, set

$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \tag{4.41}$$

$$\alpha = \operatorname{atan2}(r_{21}, r_{11}) \tag{4.42}$$

$$\gamma = \operatorname{atan2}(r_{32}, r_{33}), \tag{4.43}$$

where the square root is taken to be positive.

(ii) If $r_{31} = 1$, then $\beta = \pi/2$, and a one-parameter family of solutions for α and γ exists. One possible solution is $\alpha = 0$ and $\gamma = \text{atan2}(r_{12}, r_{22})$.

(iii) If $r_{31} = -1$, then $\beta = -\pi/2$, and a one-parameter family of solutions for α and γ exists. One possible solution is $\alpha = 0$ and $\gamma = -\text{atan2}(r_{12}, r_{22})$.

From the earlier wrist mechanism illustration of the ZYX Euler angles it should be evident that the choice of zero position for β is, in some sense, arbitrary. That is, we could just as easily have defined the home position of the wrist mechanism to be as in Figure 4.4; this would then lead to another three-parameter representation (α, β, γ) for $SO(3)$. Figure 4.4 is precisely the definition of the **ZYZ Euler angles**. The resulting rotation matrix can be obtained via the following sequence of rotations:

$$\begin{aligned} R(\alpha,\beta,\gamma) &= \text{Rot}(\hat{\text{z}},\alpha)\cdot\text{Rot}(\hat{\text{y}},\beta)\cdot\text{Rot}(\hat{\text{z}},\gamma) \\ &= \begin{bmatrix} \text{c}_\alpha & -\text{s}_\alpha & 0 \\ \text{s}_\alpha & \text{c}_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{c}_\beta & 0 & \text{s}_\beta \\ 0 & 1 & 0 \\ -\text{s}_\beta & 0 & \text{c}_\beta \end{bmatrix} \begin{bmatrix} \text{c}_\gamma & -\text{s}_\gamma & 0 \\ \text{s}_\gamma & \text{c}_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \text{c}_\alpha\text{c}_\beta\text{c}_\gamma - \text{s}_\alpha\text{s}_\gamma & -\text{c}_\alpha\text{c}_\beta\text{s}_\gamma - \text{s}_\alpha\text{c}_\gamma & \text{c}_\alpha\text{s}_\beta \\ \text{s}_\alpha\text{c}_\beta\text{c}_\gamma + \text{c}_\alpha\text{s}_\gamma & -\text{s}_\alpha\text{c}_\beta\text{s}_\gamma + \text{c}_\alpha\text{c}_\gamma & \text{s}_\alpha\text{s}_\beta \\ -\text{s}_\beta\text{c}_\gamma & \text{s}_\beta\text{s}_\gamma & \text{c}_\beta \end{bmatrix}. \end{aligned} \tag{4.44}$$

Just as before, we can show that for every rotation $R \in SO(3)$, there exists a triple (α, β, γ) that satisfies $R = R(\alpha, \beta, \gamma)$ for $R(\alpha, \beta, \gamma)$ as given in Equation (4.44). (Of course, the resulting formulas will differ from those for the ZYX Euler angles.)

From the wrist mechanism interpretation of the ZYX and ZYZ Euler angles, it should be evident that for Euler angle parametrizations of $SO(3)$, what really matters is that rotation axis 1 is orthogonal to rotation axis 2, and that rotation axis 2 is orthogonal to rotation axis 3 (axis 1 and axis 3 need not necessarily be orthogonal to each other). Specifically, any sequence of rotations of the form

$$\text{Rot}(\text{axis1},\alpha)\cdot\text{Rot}(\text{axis2},\beta)\cdot\text{Rot}(\text{axis3},\gamma), \tag{4.45}$$

where axis1 is orthogonal to axis2, and axis2 is orthogonal to axis3, can serve as a valid three-parameter representation for $SO(3)$. Later in this chapter we shall see how to express a rotation about an arbitrary axis that is not a unit axis of the reference frame.

4.2.4 Roll-Pitch-Yaw Angles

Earlier in the chapter we asserted that a rotation matrix can also be used to describe a transformation of a rigid body from one orientation to another. Here we will use this prescriptive viewpoint to derive another three-parameter representation for rotation matrices, the **roll-pitch-yaw angles**. Referring to Figure 4.5, given a frame in the identity configuration (that is, $R = I$), we first rotate this frame by an angle γ about the $\hat{\text{x}}$-axis of the fixed frame, followed by

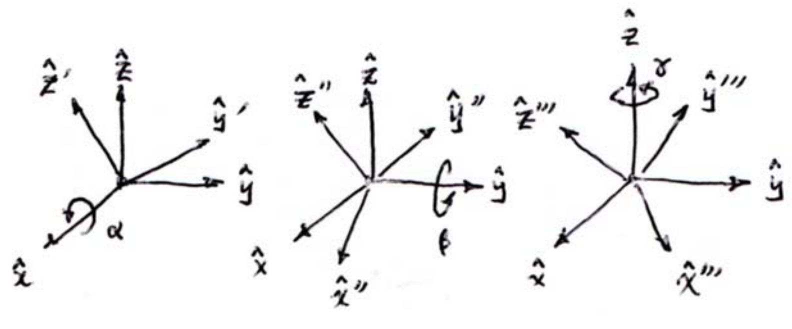

Figure 4.5: Illustration of XYZ roll-pitch-yaw angles.

an angle β about the ŷ-axis of the fixed frame, and finally by an angle α about the ẑ-axis of the fixed frame.

Let us derive the explicit form of a vector $v \in \mathbb{R}^3$ (expressed as a column vector using fixed frame coordinates) that is rotated about the fixed frame x̂-axis by an angle γ. The rotated vector, denoted v', will be

$$v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} v. \tag{4.46}$$

If v' is now rotated about the fixed frame ŷ-axis by an angle β, then the rotated vector v'' can be expressed in fixed frame coordinates as

$$v'' = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} v'. \tag{4.47}$$

Finally, rotating v'' about the fixed frame ẑ-axis by an angle α yields the vector

$$v''' = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} v''. \tag{4.48}$$

If we now take v to successively be the three unit axes of the reference frame in the identity orientation $R = I$, then after applying the above sequence of rotations to the three axes of the reference frame, its final orientation will be

$$\begin{aligned} R(\alpha,\beta,\gamma) &= \mathrm{Rot}(\hat{\mathrm{x}},\alpha)\cdot\mathrm{Rot}(\hat{\mathrm{y}},\beta)\cdot\mathrm{Rot}(\hat{\mathrm{z}},\gamma)\cdot I \\ &= \begin{bmatrix} \mathrm{c}_\alpha & -\mathrm{s}_\alpha & 0 \\ \mathrm{s}_\alpha & \mathrm{c}_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}_\beta & 0 & \mathrm{s}_\beta \\ 0 & 1 & 0 \\ -\mathrm{s}_\beta & 0 & \mathrm{c}_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_\gamma & -\mathrm{s}_\gamma \\ 0 & \mathrm{s}_\gamma & \mathrm{c}_\gamma \end{bmatrix} \cdot I \\ &= \begin{bmatrix} \mathrm{c}_\alpha\mathrm{c}_\beta & \mathrm{c}_\alpha\mathrm{s}_\beta\mathrm{s}_\gamma - \mathrm{s}_\alpha\mathrm{c}_\gamma & \mathrm{c}_\alpha\mathrm{s}_\beta\mathrm{c}_\gamma + \mathrm{s}_\alpha\mathrm{s}_\gamma \\ \mathrm{s}_\alpha\mathrm{c}_\beta & \mathrm{s}_\alpha\mathrm{s}_\beta\mathrm{s}_\gamma + \mathrm{c}_\alpha\mathrm{c}_\gamma & \mathrm{s}_\alpha\mathrm{s}_\beta\mathrm{c}_\gamma - \mathrm{c}_\alpha\mathrm{s}_\gamma \\ -\mathrm{s}_\beta & \mathrm{c}_\beta\mathrm{s}_\gamma & \mathrm{c}_\beta\mathrm{c}_\gamma \end{bmatrix}. \end{aligned} \tag{4.49}$$

This product of three rotations is exactly the same as that for the ZYX Euler angles given in (4.40). We see that the same product of three rotations admits two different physical interpretations: as a sequence of rotations with respect to the body frame (ZYX Euler angles), or, reversing the order of rotations, as a sequence of rotations with respect to the fixed frame (the XYZ roll-pitch-yaw angles).

The terms roll, pitch, and yaw are often used to describe the rotational motion of a ship or aircraft. In the case of a typical fixed-wing aircraft, for example, suppose a body frame is attached such that the $\hat{\mathrm{x}}$-axis is in the direction of forward motion, the $\hat{\mathrm{z}}$-axis is the vertical axis pointing downward toward ground (assuming the aircraft is flying level with respect to ground), and the $\hat{\mathrm{y}}$-axis extends in the direction of the wing. The roll, pitch, and yaw motions are then defined according to the XYZ roll-pitch-yaw angles (α, β, γ) of Equation (4.49).

In fact, for any sequence of rotations of the form (4.45) in which consecutive axes are orthogonal, a similar descriptive-prescriptive interpretation exists for the corresponding Euler angle formula. Euler angle formulas can be defined in a number of ways depending on the choice and order of the rotation axes, but their common features are:

- The angles represent three successive rotations taken about the axes of either the body frame or the fixed frame.
- The first axis must be orthogonal to the second axis, and the second axis must be orthogonal to the third axis.
- The angle of rotation for the first and third rotations ranges in value over a 2π interval, while that of the second rotation ranges in value over an interval of length π.

4.2.5 Exponential Coordinates

We now introduce another three-parameter representation for rotations, the **exponential coordinates**. In the Euler angle representation, a rotation matrix is expressed as a product of three rotations, each depending on a single parameter. The exponential coordinates parametrize a rotation matrix in terms of a single rotation axis (represented by a vector ω of unit length), together with an angle of rotation θ about that axis; The vector $r = \omega\theta \in \mathbb{R}^3$ then serves as a three-parameter representation of the rotation. This representation is most naturally introduced in the setting of linear differential equations, whose main results we now review.

4.2.5.1 Some Basic Results from Linear Differential Equations

Let us begin with the simplest scalar linear differential equation

$$\dot{x}(t) = ax(t), \tag{4.50}$$

where $x(t) \in \mathbb{R}$, $a \in \mathbb{R}$ is constant, and initial condition $x(0) = x_0$ is assumed given ($\dot{x}(t)$ denotes the derivative of $x(t)$ with respect to t). Equation (4.50) has solution

$$x(t) = e^{at}x_0.$$

It is also useful to remember the series expansion of the exponential function:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \ldots$$

Now consider the vector linear differential equation

$$\dot{x}(t) = Ax(t) \tag{4.51}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is constant, and initial condition $x(0) = x_0$ is given. From the earlier scalar result, one can conjecture a solution of the form

$$x(t) = e^{At}x_0 \tag{4.52}$$

where the matrix exponential e^{At} now needs to be defined in a meaningful way. Again mimicking the scalar case, we define the matrix exponential to be

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots \tag{4.53}$$

The first question to be addressed is under what conditions this series converges, so that the matrix exponential is well-defined. It can be shown that if A is constant and finite, this series is always guaranteed to converge to a finite limit; the proof can be found in most texts on ordinary linear differential equations and will not be covered here.

The second question is whether (4.52) is indeed a solution to (4.51). Taking the time derivative of $x(t) = e^{At}x_0$,

$$\begin{aligned}
\dot{x}(t) &= \left(\frac{d}{dt}e^{At}\right)x_0 \\
&= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots\right)x_0 \\
&= \left(A + A^2t + \frac{A^3t^2}{2!} + \ldots\right)x_0 \\
&= Ae^{At}x_0 \\
&= Ax(t),
\end{aligned} \tag{4.54}$$

which proves that $x(t) = e^{At}x_0$ is indeed a solution. That this is a unique solution follows from the basic existence and uniqueness result for linear ordinary differential equations, which we will just invoke here without proof.

Note that in the fourth line of (4.54), A could also have been factored to the right, i.e.,

$$\dot{x}(t) = e^{At}Ax_0.$$

While for arbitrary square matrices A and B we have in general that $AB \neq BA$, it is always true that

$$Ae^{At} = e^{At}A \tag{4.55}$$

for any square A and scalar t (you can verify this directly using the series expansion for the matrix exponential).

While the matrix exponential e^{At} is defined as an infinite series, closed-form expressions are often available. For example, if A can be expressed as $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2!} + \ldots \\ &= I + (PDP^{-1})t + (PDP^{-1})(PDP^{-1})\frac{t^2}{2!} + \ldots \\ &= P(I + Dt + \frac{(Dt)^2}{2!} + \ldots)P^{-1} \\ &= Pe^{Dt}P^{-1}. \end{aligned} \tag{4.56}$$

If moreover D is diagonal, i.e., $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$, then its matrix exponential is particularly simple to evaluate:

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}. \tag{4.57}$$

We summarize the above results in the following proposition:

Proposition 4.7. *The linear differential equation $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$, where $A \in \mathbb{R}^{n\times n}$ is constant and $x(t) \in \mathbb{R}^n$, has solution*

$$x(t) = e^{At}x_0 \tag{4.58}$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots. \tag{4.59}$$

The matrix exponential e^{At} further satisifies the following properties:

(i) $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

(ii) *If $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$, then $e^{At} = Pe^{Dt}P^{-1}$.*

(iii) *If $AB = BA$, then $e^Ae^B = e^{A+B}$.*

(iv) $(e^A)^{-1} = e^{-A}$.

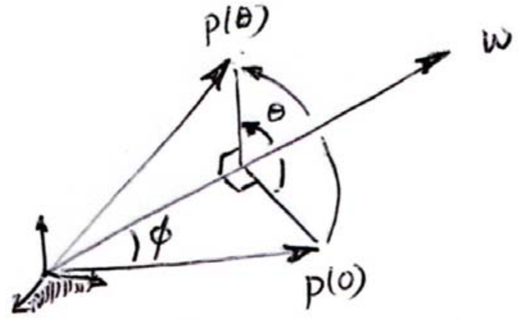

Figure 4.6: The vector $p(0)$ is rotated by an angle θ about the axis ω, to $p(\theta)$.

The third property can be established by expanding the exponentials and comparing terms. The fourth property follows by setting $B = -A$ in the third property.

A final linear algebraic result useful in finding closed-form expressions for e^{At} is the Cayley-Hamilton Theorem, which we state here without proof:

Proposition 4.8. *Let $A \in \mathbb{R}^{n \times n}$ be constant, with characteristic polynomial*

$$p(s) = \det(Is - A) = s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0.$$

Then

$$p(A) = A^n + c_{n-1}A^{n-1} + \ldots + c_1 A + c_0 I = 0.$$

4.2.5.2 Exponential Coordinates for Rotations

In the exponential coordinate representation for rotations, a rotation is represented by a single axis of rotation together with a rotation angle about the axis. Referring to Figure 4.6, suppose a three-dimensional vector $p(0)$ is rotated by an angle θ about the fixed rotation axis ω to $p(\theta)$; here we assume all quantities are expressed in fixed frame coordinates and $\|\omega\| = 1$. This rotation can be achieved by imagining that $p(0)$ is subject to a rotation about ω at a constant rate of 1 rad/sec, from time $t = 0$ to $t = \theta$. Let $p(t)$ denote this path. The velocity of $p(t)$, denoted $\dot{p}$, is then given by

$$\dot{p} = \omega \times p. \tag{4.60}$$

To see why this is true, let ϕ be the angle between $p(t)$ and ω. Observe that p traces a circle of radius $\|p\| \sin\phi$ about the ω-axis. Then $\dot{p} = \omega \times p$ is tangent to the path with magnitude $\|p\| \sin\phi$, which is exactly (4.60).

We now propose an alternative way of expressing the cross-product between two vectors. To do this we introduce the following notation:

Definition 4.3. Given a vector $\omega \in \mathbb{R}^3$, define

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{4.61}$$

The matrix $[\omega]$ as defined above is **skew-symmetric**; that is,

$$[\omega] = -[\omega]^T.$$

With this representation it is a simple calculation to verify the following property:

Proposition 4.9. *Given two vectors $x, y \in \mathbb{R}^3$, their cross product $x \times y$ can be obtained as*

$$x \times y = [x]y, \tag{4.62}$$

where $[x]$ is the skew-symmetric matrix representation of x. Also,

$$[x \times y] = [x][y] - [y][x]. \tag{4.63}$$

Another useful property involving rotations and skew-symmetric matrices is the following:

Proposition 4.10. *Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following always holds:*

$$R[\omega]R^T = [R\omega]. \tag{4.64}$$

Proof. Letting r_i^T be the ith row of R,

$$\begin{aligned} R[\omega]R^T &= \begin{bmatrix} r_1^T(\omega \times r_1) & r_1^T(\omega \times r_2) & r_1^T(\omega \times r_3) \\ r_2^T(\omega \times r_1) & r_2^T(\omega \times r_2) & r_2^T(\omega \times r_3) \\ r_3^T(\omega \times r_1) & r_3^T(\omega \times r_2) & r_3^T(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^T\omega & r_2^T\omega \\ r_3^T\omega & 0 & -r_1^T\omega \\ -r_2^T\omega & r_1^T\omega & 0 \end{bmatrix} \\ &= [R\omega], \end{aligned} \tag{4.65}$$

where the second line makes use of the determinant formula for 3×3 matrices, i.e., if M is a 3×3 matrix with columns $\{a, b, c\}$, then $\det M = a^T(b \times c) = c^T(a \times b) = b^T(c \times a)$. □

With the above results, the differential equation (4.60) can be expressed as

$$\dot{p} = [\omega]p, \tag{4.66}$$

with initial condition $p(0) = 0$. This is a linear differential equation of the form $\dot{x} = Ax$ that we have studied earlier; its solution is given by

$$p(t) = e^{[\omega]t}p(0).$$

Since t and θ are interchangeable as a result of assuming that $p(t)$ rotates at a constant rate of 1 rad/sec (after t seconds, $p(t)$ will have rotated t radians), the above can also be written

$$p(\theta) = e^{[\omega]\theta} p(0).$$

We now derive a closed-form expression for $e^{[\omega]\theta}$. Here we make use of the Cayley-Hamilton Theorem for $[\omega]$. First, the characteristic polynomial associated with the 3×3 matrix $[\omega]$ is given by

$$p(s) = \det(Is - [\omega]) = s^3 + s.$$

The Cayley-Hamilton Theorem then implies $[\omega]^3 + [\omega] = 0$, or

$$[\omega]^3 = -[\omega].$$

Let us now expand the matrix exponential $e^{[\omega]\theta}$ in series form. Replacing $[\omega]^3$ by $-[\omega]$, $[\omega]^4$ by $-[\omega]^2$, $[\omega]^5$ by $-[\omega]^3 = [\omega]$, and so on, we obtain

$$\begin{aligned} e^{[\omega]\theta} &= I + [\omega]\theta + [\omega]^2 \frac{\theta^2}{2!} + [\omega]^3 \frac{\theta^3}{3!} + \ldots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \right) [\omega] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots \right) [\omega]^2. \end{aligned}$$

Now recall the series expansions for $\sin\theta$ and $\cos\theta$:

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \end{aligned}$$

The exponential $e^{[\omega]\theta}$ therefore simplifies to the following:

Proposition 4.11. *Given a vector $\omega \in \mathbb{R}^3$ such that $\|\omega\| = 1$, and any scalar $\theta \in \mathbb{R}$,*

$$e^{[\omega]\theta} = I + \sin\theta\, [\omega] + (1 - \cos\theta)[\omega]^2. \tag{4.67}$$

This formula is also known as the **Rodrigues formula** for rotations.

4.2.5.3 Matrix Logarithm of a Rotation Matrix

So far we have shown that multiplying a vector $v \in \mathbb{R}^3$ by the exponential matrix $e^{[\omega]\theta}$ above amounts to rotating v about the axis ω by an angle θ. Our next task is to show that for any given rotation matrix $R \in SO(3)$, one can always find a unit vector ω and scalar θ such that $R = e^{[\omega]\theta}$. Defining $r = \omega\theta \in \mathbb{R}^3$, we shall call the skew-symmetric matrix $[r]$ the **matrix logarithm** of R. In this way we can assert that r provides another three-parameter representation for rotations. Expanding each of the entries for $e^{[\omega]\theta}$ in (4.67) leads to the following expression:

$$\begin{bmatrix} c_\theta + \omega_1^2(1 - c_\theta) & \omega_1\omega_2(1 - c_\theta) - \omega_3 s_\theta & \omega_1\omega_3(1 - c_\theta) + \omega_2 s_\theta \\ \omega_1\omega_2(1 - c_\theta) + \omega_3 s_\theta & c_\theta + \omega_2^2(1 - c_\theta) & \omega_2\omega_3(1 - c_\theta) - \omega_1 s_\theta \\ \omega_1\omega_3(1 - c_\theta) - \omega_2 s_\theta & \omega_2\omega_3(1 - c_\theta) + \omega_1 s_\theta & c_\theta + \omega_3^2(1 - c_\theta) \end{bmatrix}, \tag{4.68}$$

where $(\omega_1, \omega_2, \omega_3)$ are the elements of ω, and we use the shorthand notation $s_\theta = \sin\theta$, $c_\theta = \cos\theta$. Setting the above equal to the given $R \in SO(3)$ and subtracting the transpose from both sides leads to the following:

$$\begin{aligned} r_{32} - r_{23} &= 2\omega_1 \sin\theta \\ r_{13} - r_{31} &= 2\omega_2 \sin\theta \\ r_{21} - r_{12} &= 2\omega_3 \sin\theta. \end{aligned}$$

Therefore as long as $\sin\theta \neq 0$ (or equivalently, θ is not an integer multiple of π), we can write

$$\begin{aligned} \omega_1 &= \frac{1}{2\sin\theta}(r_{32} - r_{23}) \\ \omega_2 &= \frac{1}{2\sin\theta}(r_{13} - r_{31}) \\ \omega_3 &= \frac{1}{2\sin\theta}(r_{21} - r_{12}). \end{aligned}$$

The above equations can also be expressed in skew-symmetric matrix form as

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \frac{1}{2\sin\theta}\left(R - R^T\right). \tag{4.69}$$

Recall that ω represents the axis of rotation for the given R. Because of the $\sin\theta$ term in the denominator, $[\omega]$ is not well-defined if θ is an integer multiple of π. We address this situation next, but for now let us assume this is not the case and find an expression for θ. Setting R equal to (4.68) and taking the trace of both sides (recall that the trace of a matrix is the sum of its diagonals),

$$\operatorname{tr} R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta. \tag{4.70}$$

The above follows since $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$. For any θ satisfying $1 + 2\cos\theta = \operatorname{tr} R$ such that θ is not an integer multiple of π, R can be expressed as the exponential $e^{[\omega]\theta}$ with $[\omega]$ as given in (4.69).

Let us now return to the case $\theta = k\pi$, where k is some integer. When k is an even integer (corresponding to $\theta = 0, \pm 2\pi, \pm 4\pi, \ldots$) we have $\operatorname{tr} R = 3$, or equivalently $R = I$, and it follows straightforwardly that $\theta = 0$ is the only possible solution. When k is an odd integer (corresponding to $\theta = \pm\pi, \pm 3\pi, \ldots$, which in turn implies $\operatorname{tr} R = -1$), the exponential formula (4.67) simplifies to

$$R = e^{[\omega]\pi} = I + 2[\omega]^2. \tag{4.71}$$

The three diagonal terms of (4.71) can be manipulated to

$$\omega_i = \pm\sqrt{\frac{r_{ii} + 1}{2}}, \quad i = 1, 2, 3. \tag{4.72}$$

These may not always lead to a feasible unit-norm solution ω. The off-diagonal terms lead to the following three equations:

$$\begin{aligned} 2\omega_1\omega_2 &= r_{12} \\ 2\omega_2\omega_3 &= r_{23} \\ 2\omega_1\omega_3 &= r_{13}, \end{aligned} \tag{4.73}$$

From (4.71) we also know that R must be symmetric: $r_{12} = r_{21}$, $r_{23} = r_{32}$, $r_{13} = r_{31}$. Both (4.72) and (4.73) may be necessary to obtain a feasible solution. Once a solution ω has been found, then $R = e^{[\omega]k\pi}$, $k = \pm\pi, \pm 3\pi, \ldots$.

From the above it can be seen that solutions for θ exist at 2π intervals. If we restrict θ to the interval $[0, \pi]$, then the following algorithm can be used to compute the **matrix logarithm** of the rotation matrix $R \in SO(3)$:

Algorithm: Given $R \in SO(3)$, find $r = \omega\theta \in \mathbb{R}^3$, where $\theta \in [0, \pi]$ and $\omega \in \mathbb{R}^3$, $\|\omega\| = 1$, such that

$$R = e^{[r]} = e^{[\omega]\theta} = I + \sin\theta\,[\omega] + (1 - \cos\theta)[\omega]^2. \tag{4.74}$$

The skew-symmetric matrix $[r] \in \mathbb{R}^{3\times 3}$ is then said to be a **matrix logarithm** of R.

(i) If $R = I$, then $\theta = 0$ and $[r] = 0$.

(ii) If $\operatorname{tr} R = -1$, then $\theta = \pi$, and set ω to any of the three following vectors that is a feasible solution:

$$\omega = \frac{1}{\sqrt{2(1 + r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix} \tag{4.75}$$

or

$$\omega = \frac{1}{\sqrt{2(1 + r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix} \tag{4.76}$$

or

$$\omega = \frac{1}{\sqrt{2(1 + r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}. \tag{4.77}$$

(iii) Otherwise $\theta = \cos^{-1}\left(\frac{\operatorname{tr} R - 1}{2}\right) \in [0, \pi)$ and

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T). \tag{4.78}$$

The formula for the logarithm suggests a picture of the rotation group $SO(3)$ as a solid ball of radius π (see Figure 4.7): given a point $r \in \mathbb{R}^3$ in this solid ball, let $\omega = r/\|r\|$ and $\theta = \|r\|$, so that $r = \omega\theta$. The rotation matrix corresponding

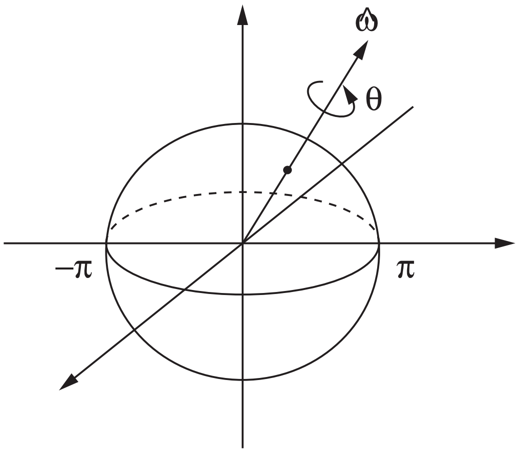

Figure 4.7: *SO(3)* as a solid ball of radius π.

to r can then be regarded as a rotation about the axis ω by an angle θ. For any $R \in SO(3)$ such that $\operatorname{tr} R \neq -1$, there exists a unique r in the interior of the solid ball such that $e^{[r]} = R$. In the event that $\operatorname{tr} R = -1$, $\log R$ is given by two antipodal points on the surface of this solid ball. That is, if if there exists some r such that $R = e^{[r]}$, then $\|r\| = \pi$, and $R = e^{[-r]}$ also holds; both r and $-r$ correspond to the same rotation R.

4.2.6 Unit Quaternions

One disadvantage with the exponential coordinates on $SO(3)$ is that because of the division by $\sin\theta$ in the logarithm formula, the logarithm can be numerically sensitive to small rotation angles θ. The **unit quaternions** are an alternative representation of rotations that alleviates some of these numerical difficulties, but at the cost of introducing an additional fourth parameter. We now illustrate the definition and use of these coordinates.

Let $R \in SO(3)$ have exponential coordinate representation $R = e^{[\omega]\theta}$, where as usual $\|\omega\| = 1$ and $\theta \in [0, \pi]$. The unit quaternion representation of R is constructed as follows. Define $q \in \mathbb{R}^4$ according to

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \omega\sin\frac{\theta}{2} \end{bmatrix} \in \mathbb{R}^4. \tag{4.79}$$

q as defined clearly satifies $\|q\| = 1$. Geometrically, q is a point lying on the three-dimensional unit sphere in $\mathbb{R}^4$, and for this reason the unit quaternions are also identified with the three-sphere, denoted S^3. Naturally among the four coordinates of q, only three can be chosen independently. Recalling that $1 + 2\cos\theta = \text{tr } R$, and using the cosine double angle formula, i.e., $\cos 2\phi = 2\cos^2\phi - 1$, the elements of q can be obtained directly from the entries of R as follows:

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} \tag{4.80}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix}. \tag{4.81}$$

Going the other way, given a unit quaternion (q_0, q_1, q_2, q_3), the corresponding rotation matrix R is obtained as a rotation about the unit axis in the direction of (q_1, q_2, q_3), by an angle $2\cos^{-1} q_0$. Explicitly,

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{4.82}$$

From the above explicit formula it should be apparent that both $q \in S^3$ and its antipodal point $-q \in S^3$ produce the same rotation matrix R. For every rotation matrix there exists two unit quaternion representations that are antipodal to each other.

The final property of the unit quaternions concerns the product of two rotations. Let $R_q, R_p \in SO(3)$ denote two rotation matrices, with unit quaternion representations $\pm q, \pm p \in S^3$, respectively. The unit quaternion representation for the product R_qR_p can then be obtained by first arranging the elements of q and p in the form of the following 2×2 complex matrices:

$$Q = \begin{bmatrix} q_0 + iq_1 & q_2 + ip_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{bmatrix}, \quad P = \begin{bmatrix} p_0 + ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & p_0 - ip_1 \end{bmatrix}, \tag{4.83}$$

where i denotes the imaginary unit. Now take the product $N = QP$, where the entries of N are written

$$N = \begin{bmatrix} n_0 + in_1 & n_2 + in_3 \\ -n_2 + in_3 & n_0 - in_1 \end{bmatrix}. \tag{4.84}$$

The unit quaternion for the product R_qR_p is then given by $\pm(n_0, n_1, n_2, n_3)$ obtained from the entries of N:

$$\begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + p_0q_1 + q_2p_3 - q_3p_2 \\ q_0p_2 + p_0q_2 - q_1p_3 + q_3p_1 \\ q_0p_3 + p_0q_3 + q_1p_2 - q_2p_1 \end{bmatrix}. \tag{4.85}$$

Verification of this formula is left as an exercise at the end of this chapter.

4.3 Rigid-Body Motions

4.3.1 Definition

We now consider representations for the combined orientation and position of a rigid body. This would seem to be fairly straightforward: once fixed and moving frames are defined, the orientation can be described by a rotation matrix $R \in SO(3)$, and the position by a vector $p \in \mathbb{R}^3$. Rather than identifying R and p separately, we shall combine them into a single matrix as follows.

Definition 4.4. The **Special Euclidean Group** $SE(3)$, also known as the group of **rigid body motions** or **homogeneous transformations** in $\mathbb{R}^3$, is the set of all 4×4 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \tag{4.86}$$

where $R \in SO(3)$, $p \in \mathbb{R}^3$, and 0 denotes the three-dimensional row vector of zeros.

An element $T \in SE(3)$ will also sometimes be denoted more compactly as $T = (R, p)$. We begin this section by establishing some basic properties of $SE(3)$, and explaining why we package R and p into this somewhat unusual matrix form.

From the definition it should be apparent that six coordinates are needed to parametrize $SE(3)$. The most obvious choice of coordinates would be to use any of the earlier three-parameter representations for $SO(3)$ (e.g., Euler angles, exponential coordinates) to parametrize the orientation R, and the usual three Cartesian coordinates in $\mathbb{R}^3$ to parametrize the position p. Instead, we shall derive a six-dimensional version of exponential coordinates on $SE(3)$ that turns out to have several advantages over these other parametrizations.

Many of the robotic mechanisms we have encountered thus far are planar. With planar rigid-body motions in mind, we make the following definition:

Definition 4.5. The **Special Euclidean Group** $SE(2)$ is the set of all 3×3 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \tag{4.87}$$

where $R \in SO(2)$, $p \in \mathbb{R}^2$, and 0 denotes the two-dimensional row vector of zeros.

A matrix $T \in SE(2)$ will always be of the form

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & p_x \\ \sin\theta & \cos\theta & p_y \\ 0 & 0 & 1 \end{bmatrix},$$

where $\theta \in [0, 2\pi]$ and $(p_x, p_y) \in \mathbb{R}^2$.

4.3.2 Properties

The following two properties of $SE(3)$ can be verified by direct calculation.

Proposition 4.12. *The product of two $SE(3)$ matrices is also an $SE(3)$ matrix.*

Proposition 4.13. *The inverse of an $SE(3)$ matrix always exists, and has the following explicit form:*

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}. \tag{4.88}$$

Before stating the next proposition, we introduce the following notation:

Definition 4.6. Given $T = (R, p) \in SE(3)$ and $x \in \mathbb{R}^3$, the product $Tx \in \mathbb{R}^3$ is then defined to be

$$Tx = Rx + p. \tag{4.89}$$

The above is a slight abuse of notation, but is motivated by the fact that if $x \in \mathbb{R}^3$ is turned into a four-dimensional vector by appending a '1', then

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}. \tag{4.90}$$

The four-dimensional vector obtained by appending a '1' to x is an example of **homogeneous coordinates**; the transformation $T \in SE(3)$ is also accordingly called a homogeneous transformation.

With the above definition of Tx, the next proposition can also be verified by direct calculation:

Proposition 4.14. *Given $T = (R, p) \in SE(3)$ and $x, y \in \mathbb{R}^3$, the following hold:*

(i) $\|Tx - Ty\| = \|x - y\|$, *where $\| \cdot \|$ denotes the standard Euclidean norm in $\mathbb{R}^3$, i.e., $\|x\| = \sqrt{x^T x}$.*

(ii) $\langle Tx - Tz, Ty - Tz \rangle = \langle x - z, y - z \rangle$ *for all $z \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^3$, i.e., $\langle x, y \rangle = x^T y$.*

In the above proposition, T is regarded as a transformation on points in $\mathbb{R}^3$, i.e., T transforms a point x to Tx. The first property then asserts that T preserves distances, while the second asserts that T preserves angles. Specifically, if $x, y, z \in \mathbb{R}^3$ represent the three vertices of a triangle, then the triangle formed by the transformed vertices $\{Tx, Ty, Tz\}$ has the same set of lengths and angles as those of the triangle $\{x, y, z\}$ (or, the two triangles are said to be isometric). One can easily imagine taking x to be the points on a rigid body, in which case Tx results in a displaced version of the rigid body. It is in this sense that $SE(3)$ can be identified with the rigid body motions.

The remaining properties describe the different physical contexts in which rigid-body motions are used. Like rotations, both descriptive and prescriptive

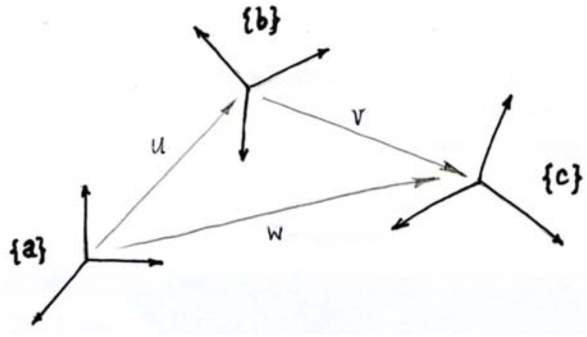

Figure 4.8: Three reference frames in space.

interpretations for rigid body motions are possible, and we begin with the former. Referring to Figure 4.8, consider three reference frames {a}, {b}, and {c} in physical space. Define u to be the vector from the origin of frame {a} to the origin of frame {b}; v and w are also defined as indicated in the figure. Denote by $T_{ab} \in SE(3)$ the position and orientation of frame {b} as seen from frame {a}; that is, R_{ab} denotes the orientation of frame {b} expressed in frame {a} coordinates, and $p_{ab} \in \mathbb{R}^3$ is the vector representation of u in $\{a\}$ frame coordinates. T_{bc} and T_{ac} are defined in a similar fashion, with $p_{ac} \in \mathbb{R}^3$ the vector representation of w in $\{a\}$ frame coordinates, and $p_{bc} \in \mathbb{R}^3$ the vector representation of **v** in $\{b\}$ frame coordinates.

Proposition 4.15. *Given three reference frames $\{a\}$, $\{b\}$, $\{c\}$ in physical space, let $T_{ab}, T_{bc}, T_{ac} \in SE(3)$ denote the relative displacements between these frames. Then $T_{ab}T_{bc} = T_{ac}$.*

Proof. Let $T_{ab}, T_{bc}, T_{ac} \in SE(3)$ be given by

$$T_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}, \quad T_{bc} = \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix}, \quad T_{ac} = \begin{bmatrix} R_{ac} & p_{ac} \\ 0 & 1 \end{bmatrix}.$$

From a previously derived property of rotation matrices we know that $R_{ab}R_{bc} = R_{ac}$. Referring to Figure 4.8, to express the vector relation u+v = w in {a} frame coordinates, we first need to find an expression for v in {a} frame coordinates. From Proposition 4.6, this is simply $R_{ab}p_{bc}$. It now follows that

$$p_{ac} = R_{ab}p_{bc} + p_{ab}. \tag{4.91}$$

The above can be combined into the following matrix equation:

$$\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ac} & p_{ac} \\ 0 & 1 \end{bmatrix}, \tag{4.92}$$

or equivalently, $T_{ab}T_{bc} = T_{ac}$. □

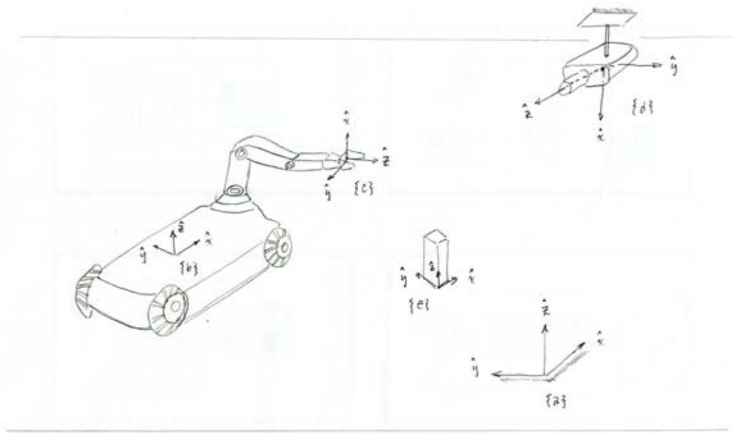

Figure 4.9: Assignment of reference frames.

The T_{ab} notation is a very convenient way of keeping track of the relationships between multiple reference frames. The following is a direct consequence of the previously established property:

Proposition 4.16. *$T_{aa} = I$, and $T_{ab}^{-1} = T_{ba}$.*

Referring again to Figure 4.8, we now change our perspective slightly, and let q denote the point in physical space corresponding to the $\{c\}$ frame origin. Let $q_b \in \mathbb{R}^3$ be the vector representation of the point q in $\{b\}$ frame coordinates; that is, $q_b = p_{bc}$. Similarly, let $q_a \in \mathbb{R}^3$ be the vector representation of the same point q in $\{a\}$ frame coordinates; that is, $q_a = p_{ac}$. Since p_{ac} and p_{bc} are related by Equation 4.91, it follows that the same relationship also exists between q_a and q_b:

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix}, \tag{4.93}$$

or in homogeneous coordinate notation,

$$q_a = T_{ab} q_b. \tag{4.94}$$

We have in fact just proven the following:

Proposition 4.17. *Given a point q in physical space, let $q_a \in \mathbb{R}^3$ and $q_b \in \mathbb{R}^3$ denote its coordinates in terms of reference frames $\{a\}$ and $\{b\}$, respectively. Then*

$$q_a = T_{ab} q_b, \tag{4.95}$$

with the product $T_{ab} q_b$ interpreted in the sense of the right-hand side of Equation (4.93).

Example

Figure 4.9 shows a robot arm mounted on a wheeled mobile platform, and a camera fixed to the ceiling. Frames {b} and {c} are respectively attached to the wheeled platform and the end-effector of the robot arm, and frame {d} is attached to the camera. A fixed frame {a} has been established, and the robot must pick up the object with body frame {e}. Suppose that the transformations T_{db} and T_{de} can be calculated from measurements obtained with the camera. The transformation T_{bc} can be determined by evaluating the forward kinematics for the current joint measurements. The transformation T_{ad} is assumed to be known in advance. Suppose these known transformations are given as follows:

$$\begin{aligned}
T_{db} &= \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_{de} &= \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_{ad} &= \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_{bc} &= \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

In order for the robot arm to pick up the object, T_{ce} must be determined. We know that

$$T_{ab}T_{bc}T_{ce} = T_{ad}T_{de},$$

where the only quantity besides T_{ce} not given to us directly is T_{ab}. However, since $T_{ab} = T_{ad}T_{de}$, we can determine T_{ce} as follows:

$$T_{ce} = (T_{ad}T_{db}T_{bc})^{-1} T_{ad}T_{de}.$$

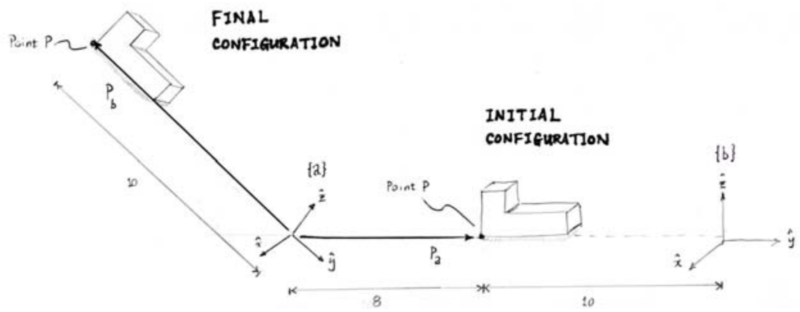

Figure 4.10: Displacement of a rigid body from an initial to final configuration.

From the given transformations,

$$T_{ad}T_{de} = \begin{bmatrix} 1 & 0 & 0 & 280 \\ 0 & 1 & 0 & -50 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ad}T_{db}T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 230 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 160 \\ 1 & 0 & 0 & 75 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(T_{ad}T_{db}T_{bc})^{-1} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 70/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 390/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

from which T_{ce} is evaluated to be

$$T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We conclude this section with an examination of the prescriptive interpretation of a rigid body motion. Figure 4.10 shows a rigid body that has been displaced from some initial configuration to a new configuration. {a} is the fixed frame, and the rigid body motion is represented by the $SE(3)$ matrix T_{ba} (we shall specify in a moment how to locate the $\{b\}$ frame from the two given configurations of the rigid body). Let p denote a point on the rigid body; $p_a \in \mathbb{R}^3$ is then the coordinates for p in the initial configuration, and $p_b \in \mathbb{R}^3$ is the coordinates for p in the displaced configuration. In terms of T_{ba} we then have the relation

$$p_b = T_{ba}p_a. \tag{4.96}$$

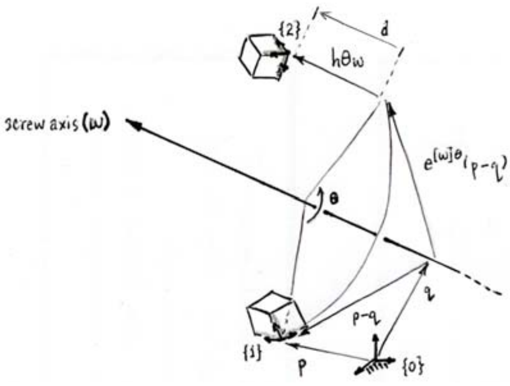

Figure 4.11: A rigid body displacement expressed as a screw motion.

The frame {b} is now drawn such that its location relative to the initial rigid body configuration is the same as the location of frame {a} relative to the displaced rigid body configuration (see Figure 4.10). For the given example, T_{ba} is given by

$$T_{ba} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & -18 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (4.97)$$

4.3.3 Screw Motions

4.3.3.1 Mathematical Representation

In the planar example given at the beginning of this chapter, we saw that any planar rigid body displacement can be achieved by rotating the rigid body about some fixed point in the plane (for a pure translation, this point lies at infinity). A similar result also exists for spatial rigid body displacements: called the **Chasles-Mozzi Theorem**, it states that every rigid body displacement can be expressed as a rotation about some fixed axis in space, followed by a pure translation parallel to that axis. In fact, switching the order of the rotation and translation will still result in the same displacement. One can therefore imagine the rotation and translation occuring simultaneously, resulting in the familiar motion of a screw.

In this section we shall develop a mathematical representation for screw motions. Figure 4.11 shows a rigid body undergoing a displacement in three-dimensional space; all vectors in the figure are expressed in terms of the fixed {0} frame coordinates. The initial and final configurations of the rigid body are labelled by frames {1} and {2}, respectively. According to the Chasles-Mozzi Theorem, there exists a screw axis—represented by the line passing through the point q and in the direction of the unit vector ω—such that the displacement can be characterized as a screw motion about this axis. The screw motion consists of a rotation about the screw axis by some angle θ, and a translation parallel to the screw axis by a distance d. As mentioned earlier, the order of the rotation and translation are interchangeable. We now derive the homogeneous transformation corresponding to this screw motion. Suppose the relative displacements T_{01} and T_{02} are given by

$$T_{01} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \tag{4.98}$$

$$T_{02} = \begin{bmatrix} R' & p' \\ 0 & 1 \end{bmatrix}. \tag{4.99}$$

The **screw pitch**, denoted h, is a scalar quantity defined as

$$h = \frac{d}{\theta}. \tag{4.100}$$

From the figure we can express R' and p' as

$$R' = e^{[\omega]\theta} R \tag{4.101}$$

$$p' = q + e^{[\omega]\theta}(p - q) + h\theta\omega. \tag{4.102}$$

The first equation is a consequence of the fact that the orientation of frame $\{2\}$ is obtained by rotating frame $\{1\}$ about the ω axis by an angle θ. The second equation follows by verifying that p' is the vectorial sum of the three vectors q, $e^{[\omega]\theta}(p - q)$, and $h\theta\omega$ as indicated in the figure. The above two equations can be combined into the following matrix equation:

$$\begin{bmatrix} R' & p' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{[\omega]\theta} & \left(I - e^{[\omega]\theta}\right) q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}. \tag{4.103}$$

The $SE(3)$ matrix

$$\begin{bmatrix} e^{[\omega]\theta} & \left(I - e^{[\omega]\theta}\right) q + h\theta\omega \\ 0 & 1 \end{bmatrix} \tag{4.104}$$

is the homogeneous transformation representation of the screw motion. In the remainder of this chapter and also the next chapter on kinematics, we shall be making the case that it is better to express this matrix as a pure matrix exponential, in the same way that a rotation matrix can be expressed as the exponential of a skew-symmetric matrix.

In the meantime we first introduce some notation.

Definition 4.7. Given $\omega, v \in \mathbb{R}^3$, let

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6, \tag{4.105}$$

which we also write more compactly as $\mathcal{S} = (\omega, v) \in \mathbb{R}^6$. The six-dimensional vector $\mathcal{S} = (\omega, v)$ is called a **twist**. Define $[S] \in \mathbb{R}^{4\times 4}$ to be the following 4×4 matrix:

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}, \quad [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{4.106}$$

where the bottom row of $[\mathcal{S}]$ consists of all zeros.

With the above notation, let us now derive a closed-form expression for the matrix exponential $e^{[\mathcal{S}]\theta}$, where $\mathcal{S} = (\omega, v)$ with $\omega \in \mathbb{R}^3$ satisfying $\|\omega\| = 1$. By an appropriate choice of v, the exponential $e^{[\mathcal{S}]\theta}$ can be made equal to the matrix of Equation (4.104); our immediate task is to find this v. Expanding the matrix exponential in series form leads to

$$\begin{aligned} e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \ldots \\ &= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \quad G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \ldots \end{aligned} \quad (4.107)$$

Noting the similarity between $G(\theta)$ and the series definition for $e^{[\omega]\theta}$, it is tempting to write $I + G(\theta)[\omega] = e^{[\omega]\theta}$, and to conclude that $G(\theta) = (e^{[\omega]\theta} - I)[\omega]^{-1}$. This is wrong: $[\omega]^{-1}$ does not exist (try computing $\det[\omega]$).

Instead we make use of the result $[\omega]^3 = -[\omega]$ that was obtained from the Cayley-Hamilton Theorem. In this case $G(\theta)$ can be simplified to

$$\begin{aligned} G(\theta) &= I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \ldots \\ &= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots\right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \ldots\right)[\omega]^2 \\ &= I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2. \end{aligned} \quad (4.108)$$

Putting everything together,

Proposition 4.18. *Let $\omega, v \in \mathbb{R}^3$ and define $\mathcal{S} = (\omega, v)$. If ω satisfies $\|\omega\| = 1$, then for any $\theta \in \mathbb{R}$,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right)v \\ 0 & 1 \end{bmatrix}. \quad (4.109)$$

If $\omega = 0$ so that $\mathcal{S} = (0, v)$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}. \quad (4.110)$$

The latter result of the propostion can be verified directly from the series expansion of $e^{[\mathcal{S}]\theta}$ with ω set to zero.

We now answer the question of what choice of v results in (4.109) equaling (4.104). The answer is

$$v = -[\omega]q + h\omega. \quad (4.111)$$

This can be verified by substituting this v into Equation (4.109) and making use of the identities $[\omega]\omega = 0$ and $[\omega]^3 = -[\omega]$. With this choice of v, the pitch h of the screw motion can then be expressed as

$$h = \omega^T v. \quad (4.112)$$

4.3.3.2 Matrix Logarithm of a Homogeneous Transformation

The above derivation essentially provides a constructive proof of the Chasles-Mozzi Theorem. That is, given an arbitrary $(R, p) \in SE(3)$, one can always find some $\mathcal{S} = (\omega, v)$ and a scalar θ such that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}. \tag{4.113}$$

In the simplest case, if $R = I$, then $\omega = 0$, and the preferred choice for v is $v = p/\|p\|$ (this would make $\theta = \|p\|$ the translation distance). If R is not the identity matrix and $\mathrm{tr}\, R \neq -1$, the solution is given by

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T) \tag{4.114}$$

$$v = G^{-1}(\theta)p, \tag{4.115}$$

where θ satisfies $1 + 2\cos\theta = \mathrm{tr}\, R$. We leave as an exercise the verification of the following formula for $G^{-1}(\theta)$:

$$G^{-1}(\theta) = \frac{1}{\theta}I + \frac{1}{2}[\omega] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\omega]^2. \tag{4.116}$$

Finally, if $\mathrm{tr}\, R = -1$, we choose $\theta = \pi$, and $[\omega]$ can be obtained via the matrix logarithm formula on $SO(3)$. Once $[\omega]$ and θ have been determined, v can then be obtained as $v = G^{-1}(\theta)p$.

Algorithm: Given $(R, p) \in SE(3)$, we seek $\mathcal{S} = (\omega, v) \in \mathbb{R}^6$ and $\theta \in [0, \pi]$ such that $e^{[\omega]\theta} = R$.

(i) If $R = I$, then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.

(ii) If $\mathrm{tr}\, R = -1$, then set $\theta = \pi$, and $[\omega] = \log R$ as determined by the matrix logarithm formula on $SO(3)$ for the case $\mathrm{tr}\, R = -1$.

(iii) Otherwise set $\theta = \cos^{-1}\left(\frac{\mathrm{tr}\, R - 1}{2}\right) \in [0, \pi)$ and

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T) \tag{4.117}$$

$$v = G^{-1}(\theta)p, \tag{4.118}$$

where $G^{-1}(\theta)$ is given by Equation (4.116).

Example

As an example, we consider the special case of planar rigid body motions and examine the matrix logarithm formula on $SE(2)$. Referring once again to Fig-

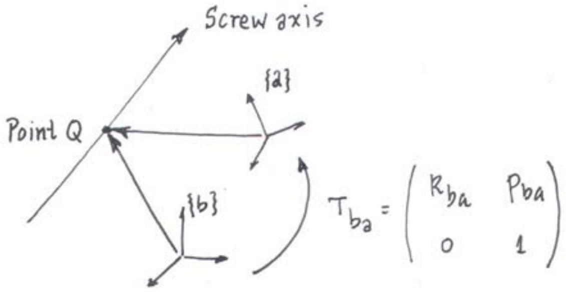

Figure 4.12: Transformation of the twist vector for a screw motion under a change of reference frames.

ure 4.1(b), suppose the initial and final configurations of the body are respectively represented by the $SE(2)$ matrices

$$
\begin{aligned}
T_{sb} &= \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 1 \\ \sin 30^\circ & \cos 30^\circ & 2 \\ 0 & 0 & 1 \end{bmatrix} \\
T_{sb} &= \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 2 \\ \sin 60^\circ & \cos 60^\circ & 1 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

For this example, the rigid body displacement occurs in the x-y plane. The corresponding screw motion will therefore have its screw axis in the direction of the z-axis, and be of zero pitch. The twist vector (ω, v) representing the screw motion will be of the form

$$
\begin{aligned}
\omega &= (0, 0, \omega_3) \\
v &= (v_1, v_2, 0).
\end{aligned}
$$

Using this reduced form, we seek the screw motion that displaces the frame at T_{sb} to T_{sc}, i.e., $T_{sc} = e^{[\mathcal{S}]\theta} T_{sb}$, or

$$T_{sc} T_{sb}^{-1} = e^{[\mathcal{S}]\theta},$$

where

$$[\mathcal{S}] = \begin{bmatrix} 0 & -\omega_3 & v_1 \\ \omega_3 & 0 & v_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can apply the matrix logarithm algorithm directly to $T_{sc} T_{sb}^{-1}$ to obtain $[\mathcal{S}]$ and θ as follows:

$$[\mathcal{S}] = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Alternatively, we can observe that the displacement is not a pure translation—T_{sb} and T_{sc} have rotation components that differ by an angle of 30°—and quickly determine that $\theta = 30°$ and $\omega_3 = 1$. We can also graphically determine the point $q = (q_x, q_y)$ on the x-y plane that the screw axis must pass through; for our example this point is given by $q = (3, 3)$. Since the screw motion has zero pitch ($h = 0$), using the relation $v = -\omega \times q + h\omega$ leads to

$$(v_x, v_y) = (q_y, -q_x) = (-3, 3).$$

4.3.3.3 Transformation Under Change of Reference Frames

We now examine how the twist vector for a screw motion transforms under a change of reference frames. For this purpose consider the general screw motion of pitch h (see Figure 4.12). Pick an arbitrary point q on this screw axis, and denote its coordinates in terms of reference frames {a} and {b} respectively by $q_a, q_b \in \mathbb{R}^3$. Let the twist for this screw motion as seen from frame {a} be given by $\mathcal{S}_a = (\omega_a, v_a)$, where $v_a = -\omega_a \times q_a + h\omega_a$. Similarly, let the twist for this same screw motion as seen from frame {b} be $\mathcal{S}_b = (\omega_b, v_b)$, where $v_b = -\omega_b \times q_b + h\omega_b$. Given the homogeneous transformation $T_{ba} = (R_{ba}, p_{ba}) \in SE(3)$, we now try to express (ω_b, v_b) in terms of (ω_a, v_a) and (R_{ba}, p_{ba}).

From our previous results we know that $\omega_b = R_{ba}\omega_a$ and $q_b = R_{ba}q_a + p_{ba}$. It follows that

$$\begin{aligned} v_b &= -R_{ba}\omega_a \times (R_{ba}q_a + p_{ba}) + hR_{ba}\omega_a \\ &= -[R_{ba}\omega_a]R_{ba}q_a - [R_{ba}\omega_a]p_{ba} + hR_{ba}\omega_a \\ &= -R_{ba}[\omega_a]R_{ba}^T R_{ba}q_a + [p_{ba}]R_{ba}\omega_a + hR_{ba}\omega_a \\ &= -R_{ba}[\omega_a]q_a + R_{ba}h\omega_a + [p_{ba}]R_{ba}\omega_a \\ &= R_{ba}(-[\omega_a]q_a + h\omega_a) + [p_{ba}]R_{ba}\omega_a \\ &= R_{ba}v_a + [p_{ba}]R_{ba}\omega_a, \end{aligned} \tag{4.119}$$

where we have made use of the properties $u \times v = [u]v$ and $R[u]R^T = [Ru]$ for any $u, v \in \mathbb{R}^3$ and $R \in SO(3)$. These equations for ω_b and v_b can be written in the equivalent matrix form

$$\begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{ba} & p_{ba} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega_a] & v_a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ba}^T & -R_{ba}^T p_{ba} \\ 0 & 1 \end{bmatrix}. \tag{4.120}$$

The above can be written as $[\mathcal{S}_b] = T_{ba}[\mathcal{S}_a]T_{ba}^{-1}$, which can be also expressed in vector form as

$$\begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R_{ba} & 0 \\ [p_{ba}]\,R_{ba} & R_{ba} \end{bmatrix} \begin{bmatrix} \omega_a \\ v_a \end{bmatrix}. \tag{4.121}$$

We introduce the following transformation to express the above relation:

Definition 4.8. Given $\mathcal{S} = (\omega, v) \in \mathbb{R}^6$, $\mathcal{S}' = (\omega', v') \in \mathbb{R}^6$, $T = (R, p) \in SE(3)$, the **Adjoint map** $\mathcal{S}' = \mathrm{Ad}_T(\mathcal{S})$ is defined as follows:

$$\mathcal{S}' = \begin{bmatrix} \omega' \\ v' \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = \mathrm{Ad}_T(\mathcal{S}). \tag{4.122}$$

The notation $[\mathrm{Ad}_T]$ is used to denote the 6×6 matrix representation of the linear transformation Ad_T:

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix}. \tag{4.123}$$

Using this notation Equation (4.122) can also be expressed as follows:

$$\mathcal{S}' = [\mathrm{Ad}_T]\mathcal{S}. \tag{4.124}$$

The adjoint map can also be equivalently expressed in the following matrix form:

$$\begin{aligned} [\mathcal{S}'] &= T[\mathcal{S}]T^{-1} \\ \begin{bmatrix} [\omega'] & v' \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} R[\omega]R^T & [p]R\omega + Rv \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.125}$$

The adjoint map satisfies the following properties, all verifiable by direct calculation:

Proposition 4.19. *Let $T_1, T_2 \in SE(3)$, and $\mathcal{S} = (\omega, v)$. Then*

$$Ad_{T_1}\left(Ad_{T_2}(\mathcal{S})\right) = Ad_{T_1 T_2}(\mathcal{S}). \tag{4.126}$$

Also, for any $T \in SE(3)$ the following holds:

$$Ad_T^{-1} = Ad_{T^{-1}}, \tag{4.127}$$

The second property follows from the first by choosing $T_1 = T^{-1}$ and $T_2 = T$, so that

$$\mathrm{Ad}_{T^{-1}}\left(\mathrm{Ad}_T(\mathcal{S})\right) = \mathrm{Ad}_{T^{-1}T}(\mathcal{S}) = \mathrm{Ad}_I(\mathcal{S}) = \mathcal{S}. \tag{4.128}$$

The following proposition states the previously derived transformation rule for the twist of a screw motion under a change of reference frames:

Proposition 4.20. *Suppose a screw motion is described in terms of reference frame $\{a\}$ by the twist $\mathcal{S}_a = (\omega_a, v_a)$, and in terms of reference frame $\{b\}$ by the twist $\mathcal{S}_b = (\omega_b, v_b)$. $\mathcal{S}_a$ and $\mathcal{S}_b$ are then related by*

$$\mathcal{S}_b = Ad_{T_{ba}}(\mathcal{S}_a) \tag{4.129}$$

$$\mathcal{S}_a = Ad_{T_{ab}}(\mathcal{S}_b). \tag{4.130}$$

We close this section by comparing the prescriptive and descriptive interpretations of a screw motion. Referring once again to Figure 4.11, the relationship between T_{01} and T_{02} can be expressed as follows:

$$T_{02} = e^{[\mathcal{S}]\theta} T_{01} \tag{4.131}$$

$$T_{01} T_{12} T_{01}^{-1} = e^{[\mathcal{S}]\theta}. \tag{4.132}$$

Here $e^{[\mathcal{S}]\theta}$ can be thought as a transformation that displaces frame T_{01} to T_{02}; this is the prescriptive interpretation of the screw motion $e^{[\mathcal{S}]\theta}$. T_{12} can also be

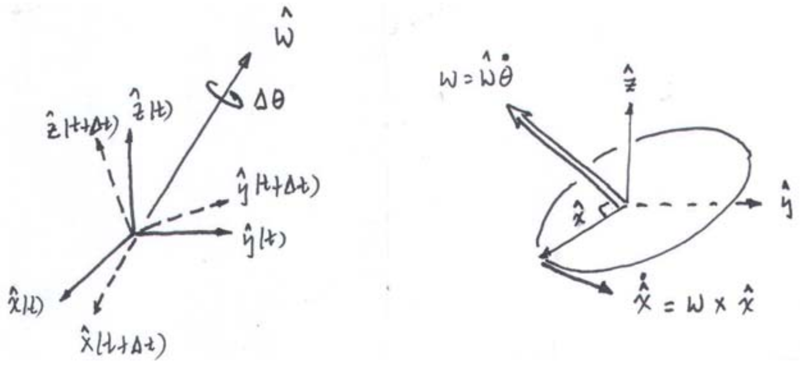

Figure 4.13: (a) The instantaneous angular velocity vector. (b) Calculating $\dot{\hat{\mathrm{x}}}$.

expressed as the matrix exponential $T_{12} = e^{[\mathcal{S}']}\theta$ for some $\mathcal{S}' = (\omega', v')$; here the screw motion $e^{[\mathcal{S}']}\theta$ is a descriptive representation for frame $\{2\}$ as seen from frame $\{1\}$. Applying the general matrix identity $Pe^{A}P^{-1} = e^{PAP^{-1}}$, we get

$$
\begin{aligned}
e^{[\mathcal{S}]\theta} &= T_{01}e^{[\mathcal{S}']\theta}T_{01}^{-1} && (4.133)\\
&= e^{\left(T_{01}[\mathcal{S}']T_{01}^{-1}\right)\theta}. && (4.134)
\end{aligned}
$$

In terms of the adjoint notation, the relation between the twists $\mathcal{S}$ and $\mathcal{S}'$ can now be expressed as

$$\mathcal{S} = \mathrm{Ad}_{T_{01}}(\mathcal{S}'). \tag{4.135}$$

Here $\mathcal{S}$ is the twist representation of the screw motion expressed in frame $\{0\}$ coordinates, while $\mathcal{S}'$ is the twist representation of the same screw motion in frame $\{1\}$ coordinates, which agrees with the previous proposition.

4.4 Velocities and Forces

4.4.1 Angular Velocities

We first examine the angular velocity of a rigid body. Referring to Figure 4.13(a), suppose a body frame with unit axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$ is attached to the rotating body. Let us determine the time derivatives of these unit axes. Beginning with $\dot{\hat{\mathrm{x}}}$, first note that $\hat{\mathrm{x}}$ is of unit length; only the direction of $\hat{\mathrm{x}}$ can vary with time (the same goes for $\hat{\mathrm{y}}$ and $\hat{\mathrm{z}}$). If we examine the body frame at times t and $t + \Delta t$—since what's relevant for us is the orientation of the body frame, for better visualization we draw the frame at the two instants with coinciding origins—the change in frame orientation from t to $t + \Delta t$ can be described as a rotation of angle $\Delta\theta$ about some unit axis $\hat{\mathrm{w}}$ passing through the origin.

In the limit as Δt approaches zero, the ratio $\frac{\Delta\theta}{\Delta t}$ becomes the rate of rotation $\dot{\theta}$, and $\hat{\mathrm{w}}$ can similarly be regarded as the instantaneous axis of rotation. In fact,

$\hat{\mathrm{w}}$ and $\dot{\theta}$ can be put together to define the **angular velocity vector** w as follows:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}. \tag{4.136}$$

It is important to note here that both the rate of rotation $\dot{\theta}$ and the rotation axis direction $\hat{\mathrm{w}}$ can vary with time. Referring to Figure 4.13(b), it should be evident that

$$\begin{aligned} \dot{\hat{\mathrm{x}}} &= \mathrm{w} \times \hat{\mathrm{x}} & (4.137)\\ \dot{\hat{\mathrm{y}}} &= \mathrm{w} \times \hat{\mathrm{y}} & (4.138)\\ \dot{\hat{\mathrm{z}}} &= \mathrm{w} \times \hat{\mathrm{z}}. & (4.139) \end{aligned}$$

Let us now express these relations explicitly in terms of fixed frame coordinates. Let $R(t)$ be the rotation matrix describing the orientation of the body frame with respect to the fixed frame. Then the first column of $R(t)$, denoted $r_1(t)$, describes $\hat{\mathrm{x}}$ in fixed frame coordinates; similarly, $r_2(t)$ and $r_3(t)$ respectively describe $\hat{\mathrm{y}}$ and $\hat{\mathrm{z}}$ in fixed frame coordinates. Let $\omega_s \in \mathbb{R}^3$ be the angular velocity w expressed in fixed frame coordinates. The previous three velocity relations expressed in fixed frame coordinates become

$$\dot{r}_i = \omega_s \times r_i = [\omega_s] r_i, \quad i = 1, 2, 3.$$

The above three equations can be rearranged into the following single matrix equation:

$$\dot{R} = [\omega_s] R. \tag{4.140}$$

Since $R^{-1} = R^T$, the above can also be written

$$\dot{R} R^T = [\omega_s]. \tag{4.141}$$

This result shows that not only is $\dot{R}R^T$ always skew-symmetric, but also admits the physical interpretation as the angular velocity in fixed-frame coordinates.

Now let ω_b be w expressed in body frame coordinates. To see how to obtain ω_b from ω_s and vice versa, let us momentarily return to our earlier notational practice of writing R as R_{sb} (recall that R_{sb} means the orientation of the body frame $\{b\}$ as seem from the fixed frame $\{s\}$). Then ω_s and ω_b are two different vector representations of the same angular velocity vector w, with $\omega_s = R_{sb}\omega_b$ and $\omega_b = R_{sb}^{-1}\omega_s$. Since $R_{sb}^{-1} = R^T$, we have

$$\omega_b = R^T \omega_s. \tag{4.142}$$

Let us now express the above relation in skew-symmetric matrix form. Recall that for any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, $R[\omega]R^T = [R\omega]$ always holds. With this result, we now express both sides of Equation (4.142) in skew-symmetric matrix form:

$$\begin{aligned} [\omega_b] &= [R^T \omega_s] \\ &= R^T [\omega_s] R \\ &= R^T (\dot{R} R^T) R \\ &= R^T \dot{R} \end{aligned} \tag{4.143}$$

In summary, we have the following two equations that relate the rotation matrix R to the angular velocity w:

Proposition 4.21. *Let $R(t)$ denote the orientation of the rotating frame as seen from the fixed frame. Denote by w the angular velocity of the rotating frame. Then*

$$\begin{aligned} \dot{R}R^{-1} &= [\omega_s] \qquad (4.144) \\ R^{-1}\dot{R} &= [\omega_b], \qquad (4.145) \end{aligned}$$

where $\omega_s \in \mathbb{R}^3$ is the fixed frame vector representation of w, and $\omega_b \in \mathbb{R}^3$ is the moving frame vector representation of w.

4.4.2 Spatial Velocities

We now consider both the linear and angular velocity of the moving frame. As before, denote by {s} and {b} the fixed (space) and moving (body) frames, respectively, and let

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix} \qquad (4.146)$$

denote the homogeneous transformation of {b} as seen from {s} (to keep the notation uncluttered, for the time being we shall write T instead of the usual T_{sb}).

In the previous section we discovered that pre- or post-multiplying $\dot{R}$ by R^{-1} results in (a skew-symmetric representation of) the angular velocity vector, either in fixed or moving frame coordinates. One might reasonably ask if a similar property carries over to $\dot{T}$, i.e., whether $T^{-1}\dot{T}$ and $\dot{T}T^{-1}$ carry similar physical interpretations.

Let us first see what happens when we pre-multiply $\dot{T}$ by T^{-1}:

$$\begin{aligned} T^{-1}\dot{T} &= \begin{bmatrix} R^T & -R^Tp \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} \qquad (4.147) \\ &= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Recall that $R^T\dot{R} = [\omega_b]$ is just the (skew-symmetric matrix representation of) the angular velocity expressed in moving frame coordinates. Also, $\dot{p}$ is the linear velocity of the moving frame origin expressed in fixed frame coordinates, and $R^T\dot{p} = v_b$ is this linear velocity expressed in moving frame coordinates. Putting these two observations together, we can conclude that $T^{-1}\dot{T}$ represents the linear and angular velocity of the moving frame in terms of the moving frame coordinates.

The previous calculation of $T^{-1}\dot{T}$ suggests that it is reasonable to merge ω_b and v_b into a single six-dimensional velocity vector. This is exactly what we shall do. Define

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}, \quad [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} = T^{-1}\dot{T}. \qquad (4.148)$$

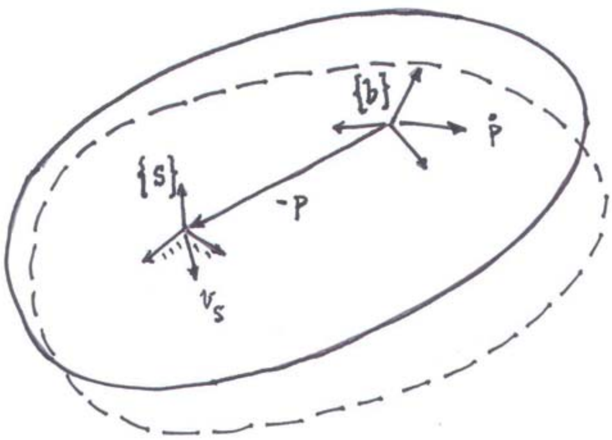

Figure 4.14: Physical interpretation of v_s. The initial (solid line) and displaced (dotted line) configurations of a rigid body.

We shall also use the more compact notation $\mathcal{V}_b = (\omega_b, v_b)$, so that $[\mathcal{V}_b]$ is the 4×4 matrix representation of $\mathcal{V}_b$. We shall call $\mathcal{V}_b$ the **spatial velocity** in the **moving** (or **body**) **frame**.

Now that we have a physical interpretation for $T^{-1}\dot{T}$, let us evaluate $\dot{T}T^{-1}$:

$$\begin{aligned} \dot{T}T^{-1} &= \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}R^T & \dot{p} - \dot{R}R^T p \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.149}$$

Observe that the skew-symmetric matrix $[\omega_s] = \dot{R}R^T$ is the angular velocity expressed in fixed frame coordinates, but that $v_s = \dot{p} - \dot{R}R^T p$ is **not** the linear velocity of the moving frame origin expressed in the fixed frame (that quantity would simply be $\dot{p}$). On the other hand, if we write v_s as

$$v_s = \dot{p} - \omega_s \times p = \dot{p} + \omega_s \times (-p), \tag{4.150}$$

the physical meaning of v_s can now be inferred: imagining the moving frame is attached to an infinitely large rigid body, v_s is the instantaneous velocity of the point on this body corresponding to the fixed frame origin (see Figure 4.14).

As we did for ω_b and v_b, we also merge ω_s and v_s into a six-dimensional velocity:

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}, \quad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1}. \tag{4.151}$$

We shall also use the more compact notation $\mathcal{V}_s = (\omega_s, v_s)$, so that $[\mathcal{V}_s]$ is the 4×4 matrix representation of $\mathcal{V}_s$. We call $\mathcal{V}_s$ the **spatial velocity** in the **fixed** (or **space**) **frame**.

If we regard the fixed and moving bodies as being infinitely large, there is in fact an appealing and natural symmetry between $\mathcal{V}_s = (\omega_s, v_s)$ and $\mathcal{V}_b = (\omega_b, v_b)$:

(i) ω_b is the angular velocity in **moving frame** coordinates;

(ii) ω_s is the angular velocity in **fixed frame** coordinates;

(iii) v_b is the linear velocity of the **moving frame origin**, in **moving frame** coordinates;

(iv) v_s is the linear velocity of the **fixed frame origin** (regarded as a point on the moving rigid body), in **fixed frame coordinates**.

$\mathcal{V}_b$ can be obtained from $\mathcal{V}_s$ as follows:

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}(\dot{T}T^{-1})T \\ &= T^{-1}[\mathcal{V}_s]\,T. \end{aligned} \tag{4.152}$$

Going the other way,

$$[\mathcal{V}_s] = T\,[\mathcal{V}_b]\,T^{-1}. \tag{4.153}$$

The reader may recognize that the relation between $\mathcal{V}_s$ and $\mathcal{V}_b$ is given precisely by the transformation rule for a twist vector of a screw motion under a change of reference frames. In fact, using the adjoint mapping Ad_T the above can be expressed as $\mathcal{V}_s = \mathrm{Ad}_T(\mathcal{V}_b)$, $\mathcal{V}_b = \mathrm{Ad}_{T^{-1}}(\mathcal{V}_s)$. This transformation rule can be more easily remembered if we write $T = T_{sb}$, in which case

$$\mathcal{V}_s = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b) \tag{4.154}$$

$$\mathcal{V}_b = \mathrm{Ad}_{T_{bs}}(\mathcal{V}_s). \tag{4.155}$$

In expanded form, the above becomes

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \tag{4.156}$$

$$\begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^T & 0 \\ -R^T\,[p] & R^T \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}. \tag{4.157}$$

The main results on spatial velocities derived thus far are summarized in the following proposition:

Proposition 4.22. *Given a fixed (space) frame* $\{s\}$ *and moving (body) frame* $\{b\}$*, let* $T_{sb}(t) \in SE(3)$ *be differentiable, where*

$$T_{sb}(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}. \tag{4.158}$$

Then

$$T_{sb}^{-1}\dot{T}_{sb} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \tag{4.159}$$

is the **spatial velocity in body coordinates**, *and*

$$\dot{T}_{sb}T_{sb}^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \tag{4.160}$$

is the **spatial velocity in space coordinates**. $\mathcal{V}_s$ *and* $\mathcal{V}_b$ *are related by*

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = Ad_{T_{sb}}(\mathcal{V}_b) \tag{4.161}$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^T & 0 \\ -R^T[p] & R^T \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = Ad_{T_{bs}}(\mathcal{V}_s). \tag{4.162}$$

4.4.3 Spatial Forces

Just as we found it advantageous to merge angular and linear velocities into a single six-dimensional spatial velocity vector, for the same reasons it will be useful to analogously merge forces and moments into a single six-dimensional **spatial force**. Toward this end, suppose a force f is being applied to a point p on a rigid body. Given some reference frame {a}, let $f_a \in \mathbb{R}^3$ denote the vector representation of f in frame {a} coordinates. This force then generates a moment with respect to the {a} frame origin; in {a} frame coordinates, this moment is

$$m_a = r_a \times f_a, \tag{4.163}$$

where $r_a \in \mathbb{R}^3$ is the vector from the {a} frame origin to p, expressed in {a} frame coordinates. We pair the force f_a and moment m_a into a single six-dimensional spatial force $\mathcal{F}_a = (m_a, f_a)$, and refer to it as the **spatial force** in frame {a} coordinates. Adopting the terminology from classical screw theory, a spatial force will also be referred to as a **wrench**.

Suppose we now wish to express the force and moment in terms of another reference frame {b}. Let $f_b \in \mathbb{R}^3$ denote the vector representation of f in {b} frame coordinates. The moment generated by this force with respect to the {b} frame origin is, again expressed in {b} frame coordinates,

$$m_b = r_b \times f_b, \tag{4.164}$$

where $r_b \in \mathbb{R}^3$ is the {b} frame representation of the vector from the {b} frame origin to p. As we did for $\mathcal{F}_a$, let us also bundle m_b and f_b into the six-dimensional spatial force $\mathcal{F}_b = (m_b, f_b)$, and refer to it as the **spatial force** in {b} frame coordinates.

We now determine the relation between $\mathcal{F}_a = (m_a, f_a)$ and $\mathcal{F}_b = (m_b, f_b)$. Referring to Figure 4.15, denote the transformation T_{ab} by

$$T_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}.$$

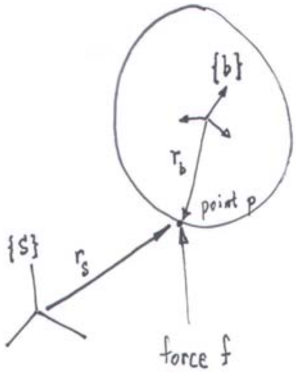

Figure 4.15: Relation between $\mathcal{F}_a$ and $\mathcal{F}_b$.

Pretty clearly $f_b = R_{ba} f_a$, which with the benefit of hindsight we shall write

$$f_b = R_{ab}^T f_a. \tag{4.165}$$

The moment m_b is given by $r_b \times f_b$, where $r_b = R_{ba}(r_a - p_{ab})$; this follows from the fact that the $r_a - p_{ab}$ is expressed in {a} frame coordinates, and must be transformed to {b} frame coordinates via multiplication by R_{ba}. Again with hindsight, we shall write

$$r_b = R_{ab}^T (r_a - p_{ab}).$$

The moment $m_b = r_b \times f_b$ can now be written in terms of f_a and m_a as

$$\begin{aligned} m_b &= R_{ab}^T (r_a - p_{ab}) \times R_{ab}^T f_a \\ &= [R_{ab}^T r_a] R_{ab}^T f_a - [R_{ab}^T p_{ab}] R_{ab}^T f_a \\ &= R_{ab}^T [r_a] f_a - R_{ab}^T [p_{ab}] f_a \\ &= R_{ab}^T m_a + R_{ab}^T [p_{ab}]^T f_a, \end{aligned} \tag{4.166}$$

where in the last line we make use of the fact that $[p_{ab}]^T = -[p_{ab}]$. Writing both m_b and f_b in terms of m_a and f_a we have, from Equations (4.165) and (4.166),

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} R_{ab} & 0 \\ [p_{ab}] R_{ab} & R_{ab} \end{bmatrix}^T \begin{bmatrix} m_a \\ f_a \end{bmatrix}, \tag{4.167}$$

or in terms of spatial forces and the adjoint map,

$$\mathcal{F}_b = \mathrm{Ad}_{T_{ab}}^T (\mathcal{F}_a) = [\mathrm{Ad}_{T_{ab}}]^T \mathcal{F}_a. \tag{4.168}$$

The above shows that under a change of reference frames, spatial velocities transform via the adjoint map, whereas spatial forces transform via the adjoint transpose map. The following proposition formally states this result.